EXERCISES FOR LECTURE 17

1. Main exercise

Exercise 1. Recall that a 2×2 matrix is nilpotent if and only if its trace and determinant are both 0. (Prove this for yourself if you didn't know this before.) Let $\mathcal{N}$ be the nilpotent cone for SL_2, so

$$\mathcal{N} = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \;\Big|\; x^2 + yz = 0 \right\}.$$

Let $R = \mathbb{C}[\mathcal{N}]$ be its coordinate ring, so that

$$R = \mathbb{C}[x, y, z]/(x^2 + yz).$$

Make R into a graded ring by declaring that x, y, and z all have degree 2. The conjugation action of SL_2 on $\mathcal{N}$ induces an action on R by ring automorphisms. Moreover, this action preserves the grading. Let

$$R\text{-mod}^{\mathrm{fg}}$$

denote the category of finitely-generated graded R-modules with a compatible SL_2-action. This is equivalent to the category

$$\mathrm{Coh}^{\mathrm{SL}_2 \times \mathbb{C}^\times}(\mathcal{N})$$

of $(\mathrm{SL}_2 \times \mathbb{C}^\times)$-equivariant coherent sheaves on $\mathcal{N}$, where $\mathbb{C}^\times$ acts on $\mathcal{N}$ by $z \cdot N = z^{-2}N$ for $N \in \mathcal{N}$.

(1) There is another description of R that is sometimes useful. Let $A = \mathbb{C}[u, v]$ be the polynomial ring in 2 variables, graded so that $\deg u = \deg v = 1$. Show that R is isomorphic to the subring $\mathbb{C}[u^2, uv, v^2] \subset A$ of polynomials with only even degree terms. Can you make SL_2 act on A by ring automorphisms in a way that extends the action on R?

(2) Let R_0 be the "trivial" R-module, i.e., the 1-dimensional vector space $R/(x, y, z)$. Let K be the kernel of $R \twoheadrightarrow R_0$.

For any $n \geq 0$, let $V(n)$ denote the irreducible SL_2-representation of highest weight n. Then one can consider the object $V(n) \otimes R \in R\text{-gmod}^{\mathrm{fg},\mathrm{SL}_2}$.

Show that $V(2)\otimes$ admits a 3-step filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset M_3 = V(2) \otimes R$$

where

$$M_1 \cong R\langle -2 \rangle, \qquad M_2/M_1 \cong K, \qquad M_3/M_2 \cong K\langle 2 \rangle.$$

(3) Show that $\mathrm{Ext}^1_R(M_3/M_1, R) \cong R_0\langle 2 \rangle$. Also describe $\mathrm{Hom}_R(M_3/M_1, R)$.

(4) Show that $R\,\mathrm{Hom}_R(R_0, R) \cong R_0[-2]\langle 2 \rangle$. Deduce that the object

$$V(n) \otimes R_0[-1]\langle 1 \rangle \in D^{\mathrm{b}} R\text{-gmod}^{\mathrm{fg},\mathrm{SL}_2}$$

is "self-dual" with respect to $R\,\mathrm{Hom}_R(-, R)$.

(5) As an R-module, A decomposes as a $A = R \oplus A^{\mathrm{odd}}$, where A^{odd} is spanned by odd-degree monomials. Show that

$$R \qquad \text{and} \qquad A^{\mathrm{odd}}$$

are also self-dual with respect to $R\operatorname{Hom}_R(-, R)$.

The self-dual objects you have determined above are precisely the simple objects in a certain abelian category

$$\mathrm{PCoh}^{\mathrm{SL}_2 \times \mathbb{C}^\times}(\mathcal{N}) \subset D^{\mathrm{b}}\mathrm{Coh}^{\mathrm{SL}_2 \times \mathbb{C}^\times}(\mathcal{N}),$$

called the category of *perverse-coherent sheaves.* They are precisely the R-modules that can arise as $H^\bullet(\mathsf{u}_q(\mathfrak{sl}_2), \text{tilting module})$.