EXERCISES FOR LECTURE 2

1. Main exercise

Exercise 1. In this exercise we take $F = \mathbb{F}_q((t))$, $\varpi = t$ and $\underline{G} = \mathrm{PGL}_2$, with the standard choices of Borel subgroup and maximal torus. Then $\underline{U}$ identifies with the subgroup of matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. In particular we have a canonical identification $\mathcal{U} = F$. We have $\mathbf{X}^\vee = \mathbb{Z}\omega^\vee$, where $\omega^\vee$ is the cocharacter

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

We have $\langle \omega^\vee, \rho \rangle = \frac{1}{2}$, and

$$r\omega^\vee \preceq s\omega^\vee \quad \text{iff} \quad s - r \in 2\mathbb{Z}_{\geq 0}.$$

(1) Without using any property stated in the lecture, show that for any $n \in \mathbb{Z}$ we have $\mathrm{d}_{\mathcal{U}}(t^n \cdot \mathbb{F}_q[[t]]) = q^{-n}$.

(2) Check that the intersections $\mathcal{U} \cap (t^{-\mu} \cdot \mathcal{K} t^\lambda \mathcal{K})$ (for $\mu \in \mathbf{X}^\vee$ and $\lambda \in \mathbf{X}^\vee_+$) are given as follows.

- If $\lambda = 2r\omega^\vee$ with $r \in \mathbb{Z}_{\geq 0}$ and if $\ell \in \mathbb{Z}$ satisfies $-r \leq \ell \leq r$, then

$$\mathcal{U} \cap (t^{-2\ell\omega^\vee} \mathcal{K} t^{2r\omega^\vee} \mathcal{K}) = \begin{cases} t^{-2r}\mathbb{F}_q[[t]] & \text{if } \ell = r; \\ \mathbb{F}_q^\times \cdot t^{-r-\ell} + t^{-r-\ell+1}\mathbb{F}_q[[t]] & \text{if } -r < \ell < r; \\ \mathbb{F}_q[[t]] & \text{if } \ell = -r. \end{cases}$$

- If $\lambda = (2r+1)\omega^\vee$ with $r \in \mathbb{Z}_{\geq 0}$ and if $\ell \in \mathbb{Z}$ satisfies $-r-1 \leq \ell \leq r$, then

$$\mathcal{U} \cap (t^{-(2\ell+1)\omega^\vee} \mathcal{K} t^{(2r+1)\omega^\vee} \mathcal{K}) = \begin{cases} t^{-2r-1}\mathbb{F}_q[[t]] & \text{if } \ell = r; \\ \mathbb{F}_q^\times \cdot t^{-\ell-r-1} + t^{-\ell-r}\mathbb{F}_q[[t]] & \text{if } -r-1 < \ell; \\ \mathbb{F}_q[[t]] & \text{if } \ell = -r-1. \end{cases}$$

Hint: You might check the containment $\supset$ explicitly in each case, considering the following products:

- $$\begin{pmatrix} t^{r-\ell} & -t^{r+\ell}Q \\ R & 0 \end{pmatrix} \cdot \begin{pmatrix} t^\ell & t^\ell Q \\ 0 & t^{-\ell} \end{pmatrix}$$
  where $Q \in \mathbb{F}_q^\times \cdot t^{-r-\ell} + t^{-r-\ell+1}\mathbb{F}_q[[t]]$ and $R \in \mathbb{F}_q[[t]]$ is the inverse to $t^{r+\ell}Q$.
- $$\begin{pmatrix} t^{r-\ell} & -t^{r+\ell+1}Q \\ R & 0 \end{pmatrix} \cdot \begin{pmatrix} t^{\ell+1} & t^{\ell+1}Q \\ 0 & t^{-\ell} \end{pmatrix}$$
  where $Q \in \mathbb{F}_q^\times \cdot t^{-\ell-r-1} + t^{-\ell-r}\mathbb{F}_q[[t]]$ and $R \in \mathbb{F}_q[[t]]$ is the inverse to $t^{r+\ell+1}Q$.

Then you might check that for any fixed ℓ, varying $r \in \mathbb{Z}_{\geq 0}$ the subsets in the right-hand side form a partition of $\mathcal{U}$, and finally conclude using the Cartan decomposition.

(3) Show that, for $\lambda \in \mathbf{X}^\vee_+$, the expansion of $\mathcal{S}(1_{\mathcal{K}t^\lambda\mathcal{K}})$ in the basis $(1_{t^\mu(\mathcal{T}\cap\mathcal{K})} : \mu \in \mathbf{X}^\vee)$ of $\mathsf{H}_\mathcal{T}$ is as follows.

- If $\lambda = 2r\omega^\vee$ with $r \in \mathbb{Z}_{\geq 0}$, then

$$\mathcal{S}(1_{\mathcal{K}t^{2r\omega^\vee}\mathcal{K}}) = q^r \cdot (1_{t^{2r\omega^\vee}(\mathcal{T}\cap\mathcal{K})} + 1_{t^{-2r\omega^\vee}(\mathcal{T}\cap\mathcal{K})}) + (q^r - q^{r-1}) \cdot \left(\sum_{-r<\ell<r} 1_{t^{2\ell\omega^\vee}(\mathcal{T}\cap\mathcal{K})} \right).$$

- If $\lambda = (2r+1)\omega^\vee$ with $r \in \mathbb{Z}_{\geq 0}$, then

$$\mathcal{S}(1_{\mathcal{K}t^{(2r+1)\omega^\vee}\mathcal{K}}) = q^{r+\frac{1}{2}} \cdot (1_{t^{(2r+1)\omega^\vee}(\mathcal{T}\cap\mathcal{K})} + 1_{t^{(-2r-1)\omega^\vee}(\mathcal{T}\cap\mathcal{K})}) + (q^{r+\frac{1}{2}} - q^{r-\frac{1}{2}}) \cdot \left(\sum_{-r-1<\ell<r} 1_{t^{(2\ell+1)\omega^\vee}(\mathcal{T}\cap\mathcal{K})} \right).$$

(4) Check explicitly that $\mathcal{S}$ satisfies the properties stated in the lecture in this case.

2. Additional exercise

Exercise 2. We continue with the setting of the preceding exercise.

(1) For $\lambda \in \mathbf{X}^\vee_+$ we set

$$M_\lambda = q^{-\langle\lambda,\rho\rangle} \cdot \sum_{\substack{\nu\in\mathbf{X}^\vee_+ \\ \nu\preceq\lambda}} 1_{\mathcal{K}t^\nu\mathcal{K}} \quad \in \mathsf{H}_\mathcal{G}.$$

Show that

$$\mathcal{S}(M_\lambda) = \sum_{\substack{\mu\in\mathbf{X}^\vee \\ -\lambda\preceq\mu\preceq\lambda}} 1_{t^\mu(\mathcal{T}\cap\mathcal{K})}.$$

(2) Check that the right-hand side is the character of the simple $\mathrm{SL}_2(\mathbb{C})$-module of highest weight λ, cf. Lecture 1. (In particular, this shows that the elements M_λ indeed correspond to those denoted similarly in the lecture.)

(3) Show using the previous formula that for $\lambda \in \mathbf{X}^\vee_+ \smallsetminus \{0\}$ we have

$$M_{\omega^\vee} \cdot M_\lambda = M_{\lambda+\omega^\vee} + M_{\lambda-\omega^\vee}.$$

(4) Prove the equalities of the previous question directly, without using the Satake isomorphism (at least for small values of λ).