EXERCISES FOR LECTURE 3

1. Main exercise

Exercise 1. In the affine Grassmannian for GL_2, consider the lattice

$$L_{(1,0)} = \mathrm{span}\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

(1) Show that the orbit $\mathrm{Gr}_{(1,0)} = \mathrm{GL}_2(\mathscr{O}) \cdot L_{(1,0)}$ is isomorphic to $\mathbb{P}^1$, via the map $L \mapsto L/t\mathscr{O}^2 \subset \mathscr{O}^2/t\mathscr{O}^2 = \mathbb{C}^2$.

(2) Show that there is a bijection

$$\mathrm{GL}_2(\mathscr{K}) \times^{\mathrm{GL}_2(\mathscr{O})} \mathrm{Gr}_{(1,0)} \cong \{(L, L') \in \mathrm{Gr} \times \mathrm{Gr} \mid tL \subset L' \subset L \text{ and } \dim L'/tL = 1\}.$$

(Here the left-hand side is the quotient of $\mathrm{GL}_2(\mathscr{K}) \times \mathrm{Gr}_{(1,0)}$ by the equivalence relation where we set $(gh, L) \sim (g, hL)$ for $g \in \mathrm{GL}_2(\mathscr{K})$, $h \in \mathrm{GL}_2(\mathscr{O})$, and $L \in \mathrm{Gr}_{(1,0)}$.)

(3) For this exercise, you may use the following claim without proof:

$$\overline{\mathrm{Gr}_{(n,0)}} = \{L \in \mathrm{Gr} \mid t^n\mathscr{O}^2 \subset L \subset \mathscr{O}^2 \text{ and } \dim L/t^n\mathscr{O}^2 = n.\}$$

Let $m : \mathrm{GL}_2(\mathscr{K}) \times^{\mathrm{GL}_2(\mathscr{O})} \mathrm{Gr}_{(1,0)} \to \mathrm{Gr}$ be the map given by $(L, L') \mapsto L'$. Consider the subset

$$\overline{\mathrm{Gr}_{(n,0)}} \mathbin{\widetilde{\times}} \mathrm{Gr}_{(1,0)} = \{(L, L') \in \mathrm{GL}_2(\mathscr{K}) \times^{\mathrm{GL}_2(\mathscr{O})} \mathrm{Gr}_{(1,0)} \mid L \in \overline{\mathrm{Gr}_{(n,0)}}\}.$$

Show that the image of $\overline{\mathrm{Gr}_{(n,0)}} \mathbin{\widetilde{\times}} \mathrm{Gr}_{(1,0)}$ under m is $\overline{\mathrm{Gr}_{(n+1,0)}}$.

(4) Determine the preimage of a point $L \in \overline{\mathrm{Gr}_{(n+1,0)}}$ under

$$m : \overline{\mathrm{Gr}_{(n,0)}} \mathbin{\widetilde{\times}} \mathrm{Gr}_{(1,0)} \to \overline{\mathrm{Gr}_{(n+1,0)}}.$$

Answer: The preimage is a single point if $L \in \mathrm{Gr}_{(n+1,0)}$, and it is isomorphic to $\mathbb{P}^1$ otherwise.