EXERCISES FOR LECTURE 4

1. Main exercises

Exercise 1. Let $G = (V, E, h, t)$ be an oriented graph, with a finite set of vertices V, and a finite set of oriented edges E with heads and tails $h, t : E \to V$.

Let us write $|G|$ for the associated topological space, i.e.,

$$(V \sqcup \bigsqcup_{e \in E} [0,1])/\sim$$

where $\sim$ is the equivalence relation identifying, for each $e \in E$, the point 0 in the associated copy of the interval $[0,1]$ with $h(e) \in V$, and 1 with $t(e) \in V$.

Write $i : V \to |G|$ for the inclusion of the vertices, and $j : U \to |G|$ for its open complement. Compute all the stalks of the sheaves $i_* \underline{\mathbb{C}}_Z$ and $j_* \underline{\mathbb{C}}_U$.

Exercise 2. Let X be the underlying topological space of a *nodal curve* over the complex numbers. I.e., one starts with an orientable smooth two dimensional compact manifold Σ, chooses a one dimensional submanifold $\Gamma \subset \Sigma$, with components $\gamma_i \simeq S^1, i \in I$, and contracts each connected component γ_i to a point. I.e.,

$$X \simeq (\Sigma \sqcup I)/\sim$$

where $\sim$ is the equivalence relation identifying all the points of γ_i with i, for $i \in I$, as in the following figure.

Denote by $j : U \to X$ the smooth locus, i.e., the image of $\Sigma \setminus \Gamma$, and compute all the stalks of the sheaf $j_* \underline{\mathbb{C}}_U$.

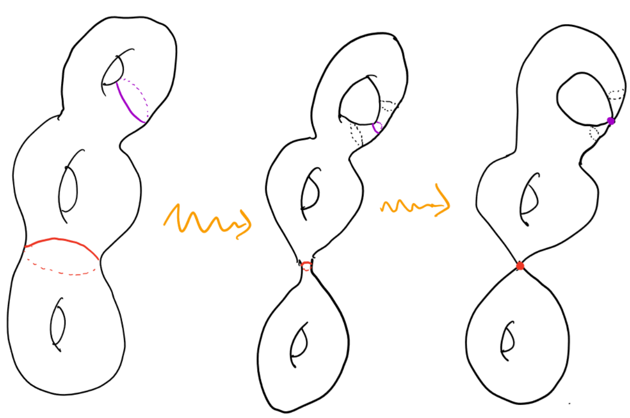

Figure 1. Degenerating from a smooth curve Σ, on the left, to a nodal curve X, on the right.

2. Additional exercise

Exercise 3. On $\mathbb{C}^\times$, consider the sub-presheaf $\mathscr{L}_{1/2}$ of the sheaf of holomorphic functions which assigns to an open subset U the vector space of holomorphic functions $\phi(z)$ on U satisfying the differential equation

$$z\partial_z(\phi) = \frac{1}{2}\phi.$$

(1) Show that $\mathscr{L}_{1/2}$ is a sheaf.
(2) Show that on a connected open $U \subset \mathbb{C}^\times$, any two sections of $\mathscr{L}_{1/2}$ differ by a scalar.
(3) Let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ be the inclusion of the punctured complex line into the complex line. Compute all the stalks of $j_*\mathscr{L}_{1/2}$.
(4) What changes in the above answers if we replace $\frac{1}{2}$ by any other $\lambda \in \mathbb{C}$?