EXERCISES FOR LECTURE 5

1. Main exercise

Exercise 1. Let $R = \mathbb{C}[x, y, z]/(x^2 + yz)$. Let R_0 denote the "trivial" R-module, the one-dimensional vector space $R/(x, y, z)$. Let M denote the R-module $R/(y, z)$.

(1) (Warmup) Find a free resolution of the trivial module $\mathbb{C}[y, z]/(y, z)$ over $\mathbb{C}[y, z]$.
(2) Find a free resolution of M over R.
(3) Compute $\operatorname{Ext}^k(M, R)$ for all $k \geq 0$. Remember, $\operatorname{Hom}(-, R)$ is contravariant!
(4) Find a short exact sequence

$$0 \to R_0 \to M \to R_0 \to 0.$$

Write down the long exact sequence for $\operatorname{Hom}(-, R)$.
(5) Deduce that $\operatorname{Ext}^1(R_0, R) = 0$.
(6) Assuming that $\operatorname{Ext}^k(R_0, R) = 0$ for $k \gg 0$, prove that $\operatorname{Ext}^2(R_0, R) = \mathbb{C}$.
(7) The ring R can be graded so that $\deg(x) = \deg(y) = \deg(z) = 2$. Repeat this exercise in the setting of graded R-modules, and deduce that

$$R \operatorname{Hom}(M, R) \cong M[-2]\langle 4 \rangle.$$

What about $R \operatorname{Hom}(R_0, R)$?

Note: It is hard to construct a free resolution of R_0, and any such resolution is infinite. For easier examples, see the supplementary exercises. For now it is more important that you can compute some Ext groups than that you can do this particular example.

2. Additional exercise

Exercise 2. Let $\mathbb{Z}$-mod$^{\text{fg}}$ denote the category of finitely generated abelian groups.

(1) Let M be an abelian group. Show that M admits a projective resolution $0 \to P_1 \to P_0 \to M \to 0$ of length at most 1. Deduce that for $i \geq 2$ and for any other abelian group N, the group $\operatorname{Ext}^i(M, N) = H^i R\operatorname{Hom}(M, N)$ vanishes.

(2) (Optional) Let M be a bounded chain complex over $\mathbb{Z}$-mod$^{\text{fg}}$. Show that there exists a quasi-isomorphism $P \to M$ where P is a bounded chain complex of *free* abelian groups.

(3) Let $F : (\mathbb{Z}\text{-mod}^{\text{fg}})^{\text{op}} \to \mathbb{Z}\text{-mod}^{\text{fg}}$ be the functor given by $F(M) = \operatorname{Hom}(M, \mathbb{Z})$. If P is a bounded chain complex of free abelian groups, show that there is a natural isomorphism $P \xrightarrow{\sim} RF(RF(P))$.

Exercise 3. Let $\Bbbk$ be a field of characteristic 2. Let $G = \mathbb{Z}/2$. This problem deals with $\Bbbk[G]$-modules.

(1) Show that $\Bbbk[G]$ is an injective module over itself. Then write down an injective resolution of the trivial $\Bbbk[G]$-module $\Bbbk$.

(2) Let $F : \Bbbk[G]\text{-mod} \to \Bbbk\text{-mod}$ be the "G-invariants" functor, given by
$$F(M) = \{m \in M \mid gm = m \text{ for all } g \in G\}.$$
Show that F is left exact. Then compute $H^i RF(\Bbbk)$ for all i.

(3) (Enrichment) Your answer to the previous question is also the singular cohomology of $\mathbb{RP}^\infty$ with coefficients in $\Bbbk$. This isn't an accident. Why are these two computations related?

Exercise 4. Practice computing Exts. Let $A = \mathbb{C}[y]/(y^4)$. For each $0 \leq i \leq 4$, let $M_i = A/(y^i)$, an i-dimensional indecomposable A-module.

(1) For each i, find a free resolution of M_i.

(2) Compute $\operatorname{Ext}^k(M_i, M_j)$ for all $k \geq 0$ and all $0 \leq i, j \leq 4$.

(3) For which i, j is $\dim \operatorname{Ext}^k(M_i, M_j) > 1$ for some $k > 0$?

Exercise 5. More practice computing Exts. Let $R = \mathbb{C}[x_1, x_2]$. Let $M = R/(x^2, xy, y^2)$.

(1) Find a free resolution of M.

(2) Let S be the R-module $R/(x, y)$. Compute $\operatorname{Ext}^k(M, S)$ for all $k \geq 0$.

(3) Compute $\operatorname{Ext}^k(M, M)$ for all $k \geq 0$.