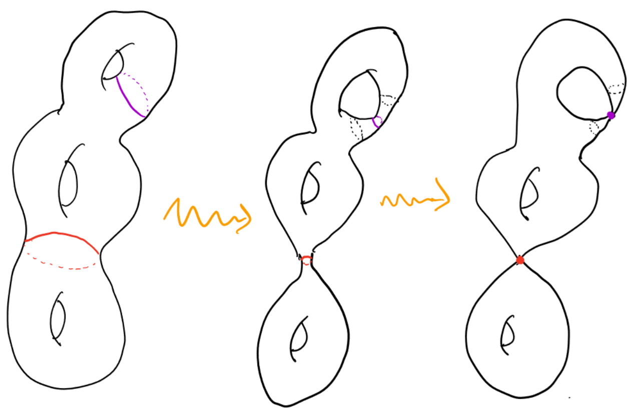
EXERCISES FOR LECTURE 6

1. Main exercise

Exercise 1. Let X be a nodal curve over $\mathbb{C}$, cf. the exercise sheet for Lecture 4, or stare again at the following picture:



Write $j : U \to X$ for its smooth locus, and $i : Z \to X$ for the complement.

(1) Compute all the $*$-stalks of $Rj_*\underline{\mathbb{C}}_U$.
(2) Using your answer to the previous part of the question, compute all the !-stalks of $\underline{\mathbb{C}}_X$.[1]
(3) Show that ω_X and $\underline{\mathbb{C}}_X[2]$ are not isomorphic in the derived category of sheaves.
(4) Using some facts from lecture, and your previous answers, compute all the !-stalks and $*$-stalks of ω_X.

[1] *Hint: use the exact triangle*

$$i_*Ri^! \to \mathrm{id} \to Rj_*j^* \xrightarrow{+1} .$$

2. Additional exercise

Exercise 2. Let X be a complex algebraic variety[2] and a pick a point $i : x \hookrightarrow X$. Generalize your answer to part (2) of the previous question as follows.

(1) Provide a canonical isomorphism between $Ri_x^! \underline{\mathbb{C}}_X$ and the shift by one of reduced cohomology of the link L_x to x, i.e.,

$$Ri_x^! \underline{\mathbb{C}}_X \simeq \varinjlim_U R\Gamma_{\mathrm{red}}(U \setminus x, \underline{\mathbb{C}}_{U \setminus x})[-1],$$

where the reduced cohomology is the cone

$$\mathbb{C} \to R\Gamma(U \setminus x, \underline{\mathbb{C}}_{U, \setminus x}) \to R\Gamma_{\mathrm{red}}(U \setminus x, \underline{\mathbb{C}}_{U \setminus x}) \xrightarrow{+1},$$

and U runs over all open neighborhoods of x.

(2) If x is a smooth point of X, where X has complex dimension d, show that any sufficiently small open U containing x satisfies $U \setminus x \simeq \mathbb{C}^d \setminus 0$ as topological spaces. Deduce a canonical isomorphism

$$Ri_x^! \underline{\mathbb{C}}_X \simeq \mathbb{C}[-2d].$$ [3]

(3) Possibly waving your hands a bit, what is the relation between your answer to (2) and the existence of a canonical isomorphism, for smooth X,

$$\omega_X \simeq \underline{\mathbb{C}}_X[2d]?$$

[2] For part (1), any sufficiently reasonable topological space will do.

[3] *Hint/warning*: if you replaced a complex manifold with a real manifold, what would fail about having such a canonical isomorphism?