EXERCISES FOR LECTURE 7

1. Main exercise

Exercise 1. Let X be a variety equipped with a stratification $(X_s)_{s\in\mathscr{S}}$, and let $\mathcal{F} \in D^{\mathrm{b}}_{\mathscr{S}}(X)$ be a constructible complex. In general, you cannot recover $\mathcal{F}$ from its table of stalks, i.e., from knowledge of the local systems $\mathcal{H}^i(\mathcal{F}|_{X_s})$ for all i and all s.

(1) Exhibit explicit examples of two nonisomorphic complexes of sheaves $\mathcal{F}$ and $\mathcal{G}$ that have the same table of stalks.

(2) Suppose you are given the additional information that $\mathcal{F}$ is a semisimple perverse sheaf. Show that in this case, $\mathcal{F}$ *is* determined by its table of stalks: in fact,

$$\mathcal{F} \cong \bigoplus_{s\in\mathscr{S}} \mathrm{IC}(X_s, \mathcal{H}^{-\dim X_s}(\mathcal{F}|_{X_s})).$$

2. Additional exercise

Exercise 2. This exercise gives more practice computing with fibers and orbits, in the setting of finite Grassmannians (this isn't related to affine Grassmannians). Let $G = GL_4$, and let Q be the subgroup of all invertible matrices of the form

$$Q = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}.$$

Then Q is the stabilizer in G of a the plane P spanned by the first basis vectors e_1 and e_2.

For any $0 \le k \le 4$, Q acts on the Grassmannian $\mathrm{Gr}(k,4)$ of k-planes (i.e. k-dimensional subspaces) in $\mathbb{C}^4$. For each $d \ge 0$, let

$$\mathrm{Gr}_d(k,4) = \{M \subset \mathbb{C}^4 \mid \dim(M) = k,\ \dim(M \cap P) = d\}.$$

(1) Prove that $\mathrm{Gr}_d(k,4)$ is preserved by the Q action, for each k and d. Indeed, these are actually the Q orbits on $\mathrm{Gr}(k,4)$.

(2) For each $0 \le k \le 4$, how does $\mathrm{Gr}(k,4)$ split into Q orbits? That is, which orbits are non-empty? Which orbits are contained in the closure of the others?

(3) Let Y be the following space:

$$Y = \{(L,M) \in \mathrm{Gr}(1,4) \times \mathrm{Gr}(2,4) \mid L \subset (M \cap P)\}.$$

What is the image of Y under the forgetful map $\pi : Y \to \mathrm{Gr}(2,4)$, and how does it decompose into Q orbits? Identify the different fibers of π. Is π semismall?

(4) What is the image of Y under the forgetful map $\rho : Y \to \mathrm{Gr}(1,4)$? What is the fiber over each point in the image? Deduce that Y is smooth, and compute its dimension.

Exercise 3. This is a continuation of the previous exercise. Let d_Y denote the complex dimension of Y.

(1) Compute the table of stalks of $R\pi_*\underline{\mathbb{C}}_Y[d_Y]$ living on $\mathrm{Gr}(2,4)$.
(2) Compute the table of stalks of $R\rho_*\underline{\mathbb{C}}_Y[d_Y]$ living on $\mathrm{Gr}(1,4)$.
(3) Compute the dimensions of all the orbits in $\mathrm{Gr}(2,4)$ and $\mathrm{Gr}(1,4)$. Note that the dimension of $\mathrm{Gr}(k,n)$ is $k(n-k)$.
(4) For each of the pushforwards above: is it perverse? If not, it is semisimple by the Decomposition theorem, so decompose it into shifts of simple perverse sheaves.

Exercise 4. Suppose G is a group acting on a variety X. Let $a : G \times X \to X$ be the action map, and let $p : G \times X \to X$ be the projection map. Recall that a G-*equivariant sheaf* on X is a sheaf $\mathcal{F}$ together with an isomorphism $\theta : a^*\mathcal{F} \cong p^*\mathcal{F}$ satisfying various conditions.

Now let $\mathcal{L}$ be a local system on G. Let $m : G \times G \to G$ be the multiplication map. The local system $\mathcal{L}$ is called *multiplicative* if $m^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$. A $(G,\mathcal{L})$-*twisted equivariant* sheaf is a sheaf $\mathcal{F}$ together with an isomorphism $\theta : a^*\mathcal{F} \cong \mathcal{L} \boxtimes \mathcal{F}$ satisfying various conditions.

(1) Let Y be another variety with a G-action, and let $f : X \to Y$ be a G-equivariant map. Show that if $(\mathcal{F}, \theta)$ is a $(G,\mathcal{L})$-equivariant sheaf on X, then the cohomology sheaves $\mathcal{H}^i(f_*\mathcal{F})$ and $\mathcal{H}^i(f_!\mathcal{L})$ admit natural $(G,\mathcal{L})$-twisted equivariant structures.
(2) Consider the group $G = \mathbb{C}^\times$, and let $\mathcal{L}$ be the rank-1 local system in which a generator of $\pi_1(\mathbb{C}^\times) = \mathbb{Z}$ acts by -1. Show that $\mathcal{L}$ is multiplicative.
(3) Let $n > 0$, and let $G = \mathbb{C}^\times$ act on $\mathbb{P}^1$ by the formula $z \cdot [a : b] = [z^n a : z^n b]$. This action has three orbits. If $\mathcal{L}$ is as before, which orbits can support a nonzero $(\mathbb{C}^\times, \mathcal{L})$-twisted equivariant sheaf? (The answer depends on n.)
(4) Let $U = \mathbb{P}^1 \smallsetminus \{0, \infty\}$, and let $j : U \to \mathbb{P}^1$ be the inclusion map. Consider the action of the preceding question with $n = 1$. Show that if $\mathcal{F}$ is a $(\mathbb{C}^\times, \mathcal{L})$-twisted equivariant sheaf on U, then $j_!\mathcal{F} = j_*\mathcal{F}$.