EXERCISES FOR LECTURE 8

1. Main exercise

Exercise 1. Let H be a complex algebraic group, and consider the tautological monoidal functor

$$\mathrm{Oblv} : \mathrm{Rep}(H) \xrightarrow{\otimes} \mathrm{Vect}_{\mathbb{C}}\,.$$

Recall the isomorphism of abstract groups $H(\mathbb{C}) \simeq \mathrm{Aut}_{\otimes}(\mathrm{Oblv})$.

(1) Call a natural transformation $\xi : \mathrm{Oblv} \to \mathrm{Oblv}$ an *infinitesimal automorphism* if it satisfies the identity

$$\xi_{V \otimes W} = \xi_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \xi_W,$$

for all $V, W \in \mathrm{Rep}(H)$. Show that if ξ and η are infinitesimal automorphisms, so is any linear combination

$$\lambda_1 \xi + \lambda_2 \eta, \qquad \lambda_1, \lambda_2 \in \mathbb{C}.$$

(2) For ξ and η as above, show that their commutator $\xi \circ \eta - \eta \circ \xi$ is again an infinitesimal automorphism. Deduce that the set of infinitesimal automorphisms naturally forms a complex Lie algebra $\mathrm{InfAut}_{\otimes}(\mathrm{Oblv})$.

(3) Writing $\mathfrak{h}$ for the Lie algebra of H, give a canonical isomorphism of Lie algebras

$$\mathfrak{h} \simeq \mathrm{InfAut}_{\otimes}(\mathrm{Oblv}).[1]$$

[1] *Hint:* Consider plugging in the left translation regular representation, i.e.,

$$\xi_{\mathcal{O}_H} : \mathcal{O}_H \to \mathcal{O}_H.$$

Argue that $\xi_{\mathcal{O}_H}$ must be (i) a derivation, with respect to the usual algebra structure on $\mathcal{O}_H$, and (ii) invariant under the right translation action. Recall (or prove, or believe) that derivations satisfying (i) and (ii) are exactly the left infinitesimal translation action of $\mathfrak{h}$ on $\mathcal{O}_H$. Finally, deduce the case of general V from that of $\mathcal{O}_H$.

2. Additional exercises

Exercise 2. (1) Let k be a field. For an algebraic group H over k, and a k-linear symmetric monoidal functor

$$\Psi : \mathrm{Rep}_k(H) \to \mathrm{Vect}_k,$$

show that $\Psi(\mathcal{O}_H)$ is naturally a commutative algebra in Vect_k with an action of H by algebra automorphisms, such that for the usual forgetful functor

$$\mathrm{Oblv} : \mathrm{Rep}_k(H) \to \mathrm{Vect}_k,$$

this recovers $\mathcal{O}_H$ with its right translation action of H.

(2) Consider the order two group $\mu_2 = \{\pm 1\}$, and its symmetric monoidal category of real representations. Check that the usual forgetful functor

$$\mathrm{Oblv} : \mathrm{Rep}_{\mathbb{R}}(\mu_2) \to \mathrm{Vect}_{\mathbb{R}}$$

sends $\mathcal{O}_{\mu_2}$ to the algebra $\mathbb{R} \times \mathbb{R}$, where $-1 \in \mu_2$ acts by swapping the two factors

$$(r_1, r_2) \mapsto (r_2, r_1).$$

(3) Show there exists a unique $\mathbb{R}$-linear symmetric monoidal functor

$$\Psi : \mathrm{Rep}_{\mathbb{R}}(\mu_2) \to \mathrm{Vect}_{\mathbb{R}}$$

which sends $\Psi(\mathcal{O}_{\mu_2})$ to the field $\mathbb{C}$, considered as an $\mathbb{R}$-algebra, with $-1 \in \mu_2$ acting by complex conjugation. Deduce that, for a general field k and group H, Oblv is not the only k-linear symmetric monoidal functor from $\mathrm{Rep}_k(H)$ to Vect.

Exercise 3. [2] Let k be a field, and H and X an algebraic group and variety over k, respectively. Consider a colimit preserving k-linear symmetric monoidal functor

$$\Xi : \mathrm{Rep}(H) \to \mathrm{QCoh}(X),$$

where the right hand side denotes quasicoherent sheaves on X, under usual tensor product.

(1) Recall or convince yourself that every finite dimensional representation V in $\mathrm{Rep}(H)$ is dualizable, in the sense of monoidal categories, with dual the usual contragredient representation. Deduce that $\Xi(V)$ is a dualizable object of $\mathrm{QCoh}(X)$.
(2) Recall or convince yourself that every dualizable object of $\mathrm{QCoh}(X)$ is a vector bundle on X. Recall or convince yourself that every object W of $\mathrm{Rep}(H)$ is the union of its finite dimensional submodules, and deduce that $\Xi(W)$ is a flat $\mathcal{O}_X$-module, which is faithfully flat if W is nonzero.
(3) Show that $\Xi(\mathcal{O}_H)$ is a faithfully flat commutative $\mathcal{O}_X$-algebra with a compatible right action of H. Denote the associated variety (technically, a priori a scheme) over X by $\mathcal{P} \to X$.

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[2] Feel extremely free to skip or skim the following exercise, especially if you have not played with faithfully flat descent before. It may also be a bit trickier than a usual exercise, so don't stress about it.

(4) Check that $\mathcal{P}$ is an H-torsor, i.e., that the natural map

$$\mathcal{P} \times H \to \mathcal{P} \underset{X}{\times} \mathcal{P}$$

is an isomorphism.[3]

(5) Check that the original symmetric monoidal functor Ξ was given by the associated bundle construction

$$\Xi(V) \simeq \mathcal{P} \overset{H}{\times} V.$$

(6) Show that the above constructions define inverse equivalences between the groupoid of symmetric monoidal functors $\Xi : \mathrm{Rep}(H) \to \mathrm{QCoh}(X)$ and the groupoid of H-torsors over X. How does this relate to Exercise 2?

[3] *Hint:* recall that, for a representation V of H, there is a canonical isomorphism of $H \times H$-representations $V \otimes \mathcal{O}_H \simeq \mathcal{O}_H \otimes V$, where the actions are given at the level of k-points by

$$(h_1, h_2) \cdot (v \otimes f) = h_1(v) \otimes h_1 \cdot f \cdot h_2, \qquad (h_1, h_2) \cdot (f \otimes v) = h_1 \cdot f \cdot h_2 \otimes h_2(v),$$

for $h_1, h_2 \in H(k), f \in \mathcal{O}_H, v \in V$.