# EXERCISES FOR LECTURE 9

## 1. Main exercise

**Exercise 1.** This problem deals with perverse sheaves on $\mathbb{P}^1$. Let $U = \mathbb{P}^1 \smallsetminus \{[1:0]\}$, and let $\mathscr{S}$ denote the stratification $\mathbb{P}^1 = \{[1:0]\} \sqcup U$. Let $j : U \to \mathbb{P}^1$ be the inclusion map.

(1) Show that $j_!\underline{\mathbb{C}}_U[1]$ is a projective object in $\mathrm{Perv}_{\mathscr{S}}(\mathbb{P}^1)$, and that $j_*\underline{\mathbb{C}}_U[1]$ is injective.

(2) On the other hand, show that $j_!\underline{\mathbb{C}}_U[1]$ is *not* injective: specifically,
$$\mathrm{Ext}^1(j_!\underline{\mathbb{C}}_U[1], \mathrm{IC}(U)) \cong \mathbb{C}.$$
Therefore, there exists a perverse sheaf $\mathcal{T}$ that fits into a nonsplit short exact sequence
$$0 \to \mathrm{IC}(U) \to \mathcal{T} \to j_!\underline{\mathbb{C}}_U[1] \to 0.$$
Moreover, $\mathcal{T}$ is unique up to isomorphism.

(3) Find a composition series for $\mathcal{T}$. What are its composition factors?

(4) Compute the table of stalks for $\mathcal{T}$ and for $\mathbb{D}\mathcal{T}$. (You should get the same answers for both.)

(5) Let $\mathcal{F}$ be any perverse sheaf with the following property: its table of stalks and that of $\mathbb{D}\mathcal{F}$ both agree with that of $\mathcal{T}$. Show that
$$\mathrm{Hom}(\mathcal{F},\mathcal{T}) \to \mathrm{Hom}(\mathcal{F}|_U, \mathcal{T}|_U) \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{T},\mathcal{F}) \to \mathrm{Hom}(\mathcal{T}|_U, \mathcal{F}|_U)$$
are both surjective. (Hint: use the distinguished triangle $i_*i^!\mathcal{T} \to \mathcal{T} \to j_*j^*\mathcal{T}_U \to$.) Deduce as a special case that the map $\mathrm{End}(\mathcal{T}) \to \mathrm{End}(\mathcal{T}|_U)$ is surjective.

(6) Compute $\mathrm{End}(\mathcal{T})$ as a ring. (Answer: it is 2-dimensional, and isomorphic to $\mathbb{C}[u]/(u^2)$.)

(7) Show that $\mathcal{T}$ is both projective and injective as an object in $\mathrm{Perv}_{\mathscr{S}}(\mathbb{P}^1)$.

## 2. Additional exercise

**Exercise 2.** This problem deals with perverse sheaves on $\mathbb{C}$ stratified by $\mathbb{C} = \{0\} \sqcup \mathbb{C}^\times$.

(1) Show that $\mathrm{Ext}^1(\underline{\mathbb{C}}_{\mathbb{C}^\times}, \underline{\mathbb{C}}_{\mathbb{C}^\times}) \cong \mathbb{C}$, so there is a nonsplit short exact sequence

$$0 \to \underline{\mathbb{C}}_{\mathbb{C}^\times} \to \mathcal{L} \to \underline{\mathbb{C}}_{\mathbb{C}^\times} \to 0.$$

Here $\mathcal{L}$ is a locally constant sheaf, but *not* a constant sheaf.

(2) Show that there is an indecomposable perverse sheaf $\mathcal{F}$ with the following table of stalks:

| | $\mathbb{C}^\times$ | $\{0\}$ |
|---|---|---|
| 0 | | |
| $-1$ | $\mathcal{L}$ | $\underline{\mathbb{C}}_{\{0\}}$ |

Find a composition series for $\mathcal{F}$.