LECTURE 1: COMBINATORICS AND REPRESENTATION THEORY OF REDUCTIVE ALGEBRAIC GROUPS

1. Reductive algebraic groups

1.1. **Subgroups.** Let $\Bbbk$ be an algebraically closed field. We will work with the following notions:

- G: connected reductive algebraic group over $\Bbbk$.
- B: Borel subgroup.
- T: maximal torus contained in B.
- U: unipotent radical of B.

We will not define these things properly. One should keep in mind the following examples.

Example 1.1. (1) $G = \mathrm{GL}_n(\Bbbk)$. One can choose for B the subgroup of upper triangular matrices and for T the subgroup of diagonal matrices. Then U is the subgroup of *unipotent* upper triangular matrices.
(2) $G = \mathrm{SL}_n(\Bbbk)$. One can choose for B and T the intersections of the previous subgroups with $\mathrm{SL}_n(\Bbbk)$.
(3) $G = \mathrm{Sp}_{2n}(\Bbbk)$ or $\mathrm{SO}_m(\Bbbk)$.

1.2. **Characters and cocharacters.** The multiplicative group $\mathbb{G}_\mathrm{m}$ is the group $\mathrm{GL}_1(\Bbbk)$. We will denote by $\mathbf{X} = X^*(T)$ the group of morphisms of algebraic groups from T to $\mathbb{G}_\mathrm{m}$, and by $\mathbf{X}^\vee = X_*(T)$ the group of morphisms of algebraic groups from $\mathbb{G}_\mathrm{m}$ to T. These are free $\mathbb{Z}$-modules of finite rank, and we have a canonical perfect pairing

$$\mathbf{X} \times \mathbf{X}^\vee \to \mathbb{Z}$$

given by composition of morphisms, based on the fact that the group of algebraic group morphisms from $\mathbb{G}_\mathrm{m}$ to $\mathbb{G}_\mathrm{m}$ identifies with $\mathbb{Z}$ via $n \leftrightarrow (z \mapsto z^n)$. Elements of $\mathbf{X}$ are called *characters*, and those of $\mathbf{X}^\vee$ are called *cocharacters*.

Example 1.2. (1) When $G = \mathrm{GL}_n(\Bbbk)$, with T as above, we have canonical identifications $\mathbf{X} = \mathbb{Z}^n$ and $\mathbf{X}^\vee = \mathbb{Z}^n$. Here $(\lambda_1, \cdots, \lambda^n)$ corresponds to the character

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto \prod_i (t_i)^{\lambda_i}$$

and to the cocharacter

$$z \mapsto \mathrm{diag}(z^{\lambda_1}, \ldots, z^{\lambda_n}).$$

We will denote by $(\varepsilon_1, \ldots, \varepsilon_n)$ the canonical basis of $\mathbb{Z}^n$ identified with $\mathbf{X}$, and by $(\delta_1, \ldots, \delta_n)$ the canonical basis of $\mathbb{Z}^n$ identified with $\mathbf{X}^\vee$.
(2) When $G = \mathrm{SL}_n(\Bbbk)$, with T as above, we have canonical identifications

$$\mathbf{X} = \mathbb{Z}^n / \Delta\mathbb{Z}$$

and

$$\mathbf{X}^\vee = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \sum_i \lambda_i = 0\}.$$

1.3. **Roots.** Let $\mathfrak{g}$, resp. $\mathfrak{b}$, be the Lie algebra of G, resp. B (i.e. the tangent space at the unit element). The action of G on itself by conjugation induces an action on $\mathfrak{g}$. It is a basic fact that the restriction to T is semisimple, i.e. we have a decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbf{X}} \mathfrak{g}_\lambda \quad \text{with } \mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \forall t \in T,\, t \cdot x = \lambda(t)x\}.$$

The *root system* is

$$\mathfrak{R} = \{\lambda \in \mathbf{X} \smallsetminus \{0\} \mid \mathfrak{g}_\lambda \neq 0\} \subset \mathbf{X}.$$

The *positive roots* is the subset

$$\mathfrak{R}_+ = \{\lambda \in \mathbf{X} \smallsetminus \{0\} \mid \mathfrak{b}_\lambda \neq 0\}.$$

The *simple roots* $\mathfrak{R}_s \subset \mathfrak{R}_+$ are the positive roots that cannot be written as the sum of several positive roots.

There is also a subset $\mathfrak{R}^\vee \subset \mathbf{X}^\vee$ of *coroots*, with a bijection $\mathfrak{R} \xrightarrow{\sim} \mathfrak{R}^\vee$. (This construction is more technical, and will not be explained.)

Example 1.3. If $G = \mathrm{GL}_n(\Bbbk)$, with the choices of B and T as above we have

$$\begin{aligned} \mathfrak{R} &= \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}, \\ \mathfrak{R}_+ &= \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}, \\ \mathfrak{R}_s &= \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < n\}. \end{aligned}$$

In fact, if $i \neq j$ the subspace $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ is the line spanned by the matrix $E_{i,j}$ with coefficient 1 is place (i,j) and 0's elsewhere.

1.4. **Weyl group.** The associated *Weyl group* is

$$W = N_G(T)/T.$$

There exist natural actions of W on $\mathbf{X}$ and $\mathbf{X}^\vee$, which are faithful. Seen as a subgroup of the group of (linear) automorphisms of $\mathbf{X}$, W contains (and, in fact, is generated by) the elements $(s_\alpha : \alpha \in \mathfrak{R})$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Example 1.4. If $G = \mathrm{GL}_n(\Bbbk)$, with the choices of B and T as above we have a canonical identification $W = \mathfrak{S}_n$. (Think of permutation matrices.)

1.5. **Bruhat decomposition.** The group W can be seen in geometry as follows. Consider the quotient G/B. This set has a canonical structure of algebraic variety over $\Bbbk$ (it is smooth and projective). For any $w \in W$ and any choice of preimage $\dot{w}$ of w in $N_G(T)$ we can consider the double coset $B\dot{w}B \subset G$; it does not depend on the choice of $\dot{w}$ hence is denoted BwB. Then the *Bruhat decomposition* says that

$$G = \bigsqcup_{w \in W} BwB,$$

hence

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

The subset $BwB/B \subset G/B$ is a locally closed algebraic subvariety, and it is isomorphic to an affine space.

Example 1.5. In case $G = \mathrm{GL}_n(\Bbbk)$ with B and T as above, the flag variety is the variety of flags of linear subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \Bbbk^n$$

with $\dim(V_i) = i$.

The Bruhat decomposition is the familiar statement that any invertible matrix M can be written as a product $M = M_1 M_2 M_3$ where M_1, M_3 are upper triangular matrices and M_2 is a permutation matrix, and that moreover M_2 is uniquely determined.

1.6. **Affine Weyl group.** The *extended affine Weyl group* is the semidirect product

$$W_{\mathrm{ext}} = W \ltimes \mathbf{X}^\vee.$$

It acts naturally on $\mathbf{X}^\vee$ by affine transformations, via

$$(wt_\lambda) \cdot \mu = w(\mu + \lambda)$$

for $w \in W$ and $\lambda, \mu \in \mathbf{X}^\vee$.

1.7. **Langlands duality.** The *root datum* of G is the quadruple $(\mathbf{X}, \mathfrak{R}, \mathbf{X}^\vee, \mathfrak{R}^\vee)$ together with the perfect pairing between $\mathbf{X}$ and $\mathbf{X}^\vee$ and the bijection between $\mathfrak{R}$ and $\mathfrak{R}^\vee$. It turns out that this datum does not depend (up to isomorphism) on the choices made above, and that G is characterized up to isomorphism by it.

There is an obvious "duality" on root data, given by

$$(\mathbf{X}, \mathfrak{R}, \mathbf{X}^\vee, \mathfrak{R}^\vee) \leftrightarrow (\mathbf{X}^\vee, \mathfrak{R}^\vee, \mathbf{X}, \mathfrak{R}).$$

Two connected reductive groups (possibly over different base fields) are called *Langlands dual* if their root data are dual. If G and $G^\vee$ are Langlands dual, then we have maximal tori $T \subset G$ and $T^\vee \subset G^\vee$ such that

$$X_*(T) = X^*(T^\vee).$$

In this case, the Weyl groups of G and $G^\vee$ identify, in such a way that the action of the Weyl group of G on $X_*(T)$ coincides with the action of Weyl group of $G^\vee$ on $X^*(T^\vee)$.

Example 1.6.
1. GL_n is Langlands self dual. The groups SL_n and PGL_n are Langlands dual.
2. SO_{2n+1} and Sp_{2n} are Langlands dual.
3. SO_{2n} is Langlands self dual.

2. Representations

2.1. **Definition.** Let G be an algebraic group over $\Bbbk$. A (finite-dimensional, algebraic) representation of G is a pair (V, ϱ) where V is a finite-dimensional $\Bbbk$-vector space and $\varrho : G \to \mathrm{GL}(V)$ is a morphism of algebraic groups.

For any affine algebraic variety X over $\Bbbk$ one can consider the $\Bbbk$-algebra $\mathcal{O}(X)$ of algebraic functions on X. In practice, an affine variety admits a description as the subset of the affine space $\mathbb{A}^n$ determined by the equations

$$P_1(x) = P_2(x) = \cdots = P_r(x) = 0$$

for some $n \geq 1$ and some polynomials $P_1, \cdots, P_r \in \Bbbk[X_1, \cdots, X_n]$, and then

$$\mathcal{O}(X) = \Bbbk[X_1, \cdots, X_n]/\sqrt{(P_1, \cdots, P_r)}.$$

Given two affine algebraic varieties X, Y, we have

$$\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes \mathcal{O}(Y)$$

where the map $\mathcal{O}(X) \to \mathcal{O}(X \times Y)$ is composition with the projection map $X \times Y \to X$, and similarly for the map $\mathcal{O}(Y) \to \mathcal{O}(X \times Y)$.

In particular, consider the algebra $\mathcal{O}(G)$ of algebraic functions from G to $\Bbbk$. This algebra has another operation (called "comultiplication")

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$

which is the result on functions of the multiplication operation $G \times G \to G$, and which makes it a *Hopf algebra*. Then the datum of a representation of G is equivalent to that of a finite-dimensional comodule for this Hopf algebra, i.e. a finite-dimensional vector space V endowed with a "coaction"

$$\Delta_V : V \to V \otimes \mathcal{O}(G)$$

which is compatible with Δ in a natural sense. In fact, $V \otimes \mathcal{O}(G)$ identifies with the space of algebraic functions from G to V, and Δ_V corresponds to $v \mapsto (g \mapsto \varrho(g)(v))$.

Example 2.1.

(1) When $G = (\mathbb{G}_{\mathrm{m}})^n$, we have $\mathcal{O}(G) = \Bbbk[x_i^{\pm 1} : 1 \leq i \leq n]$ with the comultiplication determined by

$$\Delta(x_i) = x_i \otimes x_i.$$

(For that, we identify $(\mathbb{G}_{\mathrm{m}})^n$ with the subset of $\Bbbk^{2n}$ determined by the equations $X_i X_{n+i} = 1$ for $i \in \{1, \dots, n\}$.)

(2) When $G = \mathrm{GL}_n$ we have

$$\mathcal{O}(G) = \Bbbk[x_{i,j} : 1 \leq i, j \leq n][\det^{-1}],$$

with the comultiplication determined by

$$\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}.$$

(Here we identify GL_n with the subset of $\mathrm{M}_n(\Bbbk) \times \mathbb{A}^1$ determined by the equation $\det(M) \cdot X = 1$.) Then vector space $V = \Bbbk^n$ has a canonical structure of representation GL_n. The coaction is given by

$$\Delta(e_i) = \sum_k e_k \otimes x_{k,i}$$

where $(e_1, \dots, e_n)$ is the canonical basis of $\Bbbk^n$.

The category of representations of G will be denoted $\mathrm{Rep}(G)$.

2.2. Chevalley's theorem.

The subset of *dominant weights* $\mathbf{X}^+ \subset \mathbf{X}$ is defined by

$$\mathbf{X}^+ = \{\lambda \in \mathbf{X} \mid \forall \alpha \in \mathfrak{R}_+, \ \langle \lambda, \alpha^\vee \rangle \geq 0\}.$$

(This subset is a system of representatives for the W-orbits in $\mathbf{X}$.)

Theorem 2.2 (Chevalley's theorem)**.**

(1) There exists a canonical bijection between $\mathbf{X}^+$ and the set of isomorphism classes of simple objects in $\mathrm{Rep}(G)$.

(2) If $\mathrm{char}(\Bbbk) = 0$*, then the category* $\mathrm{Rep}(G)$ *is semisimple, i.e. every representation is a direct sum of simple representations.*

Example 2.3. Consider the case $G = \mathrm{SL}_2(\Bbbk)$. We have a canonical identification $\mathbf{X} = \mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the morphism

$$\mathrm{diag}(z, z^{-1}) \mapsto z^n.$$

Under this identification, $\mathbf{X}^+ = \mathbb{Z}_{\geq 0}$.

We have a natural action on $\Bbbk^2$, hence on the space

$$\Bbbk[x, y]$$

of algebraic functions on $\Bbbk^2$. For any $n \in \mathbb{Z}_{\geq 0}$, the subspace

$$V_n = \Bbbk_n[x, y]$$

of homogeneous polynomials of degree n is stable. In case $\mathrm{char}(\Bbbk) = 0$, V_n is simple and the bijection of Chevalley's theorem is given by

$$n \mapsto V_n.$$

2.3. **Grothendieck group.** Consider the Grothendieck group $\mathsf{K}^0(\mathrm{Rep}(G))$. It is a basic fact that any representation of T is a sum of its *weight spaces*: for any $V \in \mathrm{Rep}(T)$ we have

$$V = \bigoplus_{\lambda \in X^*(G)} V_\lambda \quad \text{where } V_\lambda = \{v \in V \mid \in T,\, t \cdot v = \lambda(t)v\}.$$

This applies in particular to (restrictions of) representations of G. We therefore have a canonical "character" morphism

$$\mathrm{ch} : \mathsf{K}^0(\mathrm{Rep}(G)) \to \mathbb{Z}[\mathbf{X}]$$

given by

$$\mathrm{ch}([V]) = \sum_{\lambda \in \mathbf{X}} \dim(V_\lambda) \cdot e^\lambda.$$

Corollary 2.4. *The morphism* ch *induces an isomorphism*

$$\mathsf{K}^0(\mathrm{Rep}(G)) \to (\mathbb{Z}[\mathbf{X}])^W$$

where W acts on the right-hand side with its action on $\mathbf{X}$.

Sketch of proof. Using the action of $N_G(T) \subset G$ on representations one sees that our map factors through $(\mathbb{Z}[\mathbf{X}])^W$. Then, by Chevalley's theorem $\mathsf{K}^0(\mathrm{Rep}(G))$ admits a $\mathbb{Z}$-basis consisting of classes of simple modules. One easily sees that the coefficients of the images of these classes in the basis

$$\left(\sum_{\mu \in W\lambda} e^\mu : \lambda \in \mathbf{X}^+ \right)$$

of $(\mathbb{Z}[\mathbf{X}])^W$ is lower triangular with 1's on the diagonal for an appropriate order, which implies that these images form a basis, hence that our morphism is an isomorphism. □

Remark 2.5. The tensor product of representations equips $\mathsf{K}^0(\mathrm{Rep}(G))$ with the structure of a ring. For this structure, the isomorphism of Corollary 2.4 is a *ring* isomorphism.

3. Complement on the Weyl group (probably not covered in the lecture)

Setting

$$S = \{s_\alpha : \alpha \in \mathfrak{R}_s\},$$

it turns out that (W, S) is a *Coxeter system*; this means that W admits a presentation with generators S, and the following relations:

- $s^2 = 1$ for all $s \in S$;
- $\underbrace{st\cdots}_{m_{s,t} \text{ terms}} = \underbrace{ts\cdots}_{m_{s,t} \text{ terms}}$ for all $s \neq t \in S$

for some integers $m_{s,t} \geq 2$. (In fact, $m_{s,t}$ is the order of st in W.) Associated with this structure we have a "length function"

$$\ell : W \to \mathbb{Z}_{\geq 0}$$

such that $\ell(w)$ is the smallest integer r such that there exists a sequence $s_1, \ldots, s_r$ of elements of S such that $w = s_1 \cdots s_r$.

Example 3.1. If $G = \mathrm{GL}_n(\Bbbk)$, with the choices of B and T as above we have explained that $W = \mathfrak{S}_n$. Via this identification we have

$$S = \{(i, i+1) : 1 \leq i < n\}.$$

Moreover, $\ell(w)$ is the number of inversions of w.

This structure appears in the Bruhat decomposition (see §1.5), since the dimension of the affine space BwB/B is $\ell(w)$.

The *root lattice* $\mathbb{Z}\mathfrak{R}$ is the $\mathbb{Z}$-submodule of $\mathbf{X}$ generated by $\mathfrak{R}$. It admits a basis consisting of the simple roots. The *affine Weyl group* is the subgroup

$$W_{\mathrm{aff}} = W \ltimes \mathbb{Z}\mathfrak{R}$$

of W_{ext}. It turns out that W_{aff} is generated by the elements of the form $t_{n\alpha} s_\alpha$ for $\alpha \in \mathfrak{R}$ and $n \in \mathbb{Z}$. There is a subset $S_{\mathrm{aff}} \subset W_{\mathrm{aff}}$ containing S and such that $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system, and the associated length function extends to a function $W_{\mathrm{ext}} \to \mathbb{Z}_{\geq 0}$.

Example 3.2. If $G = \mathrm{GL}_n(\Bbbk)$, with the choices of B and T as above we have

$$S_{\mathrm{aff}} = S \sqcup \{t_{\varepsilon_1 - \varepsilon_n} s_{\varepsilon_1 - \varepsilon_n}\}.$$