LECTURE 10: GEOMETRIC SATAKE EQUIVALENCE, I

1. Geometry

1.1. **The affine Grassmannian.** Recall from Lecture 3 that we consider a complex reductive algebraic group G, its loop group $G(\mathscr{K})$, and the arc group (or positive loop group) $G(\mathscr{O})$. The associated *affine Grassmannian* is the quotient

$$\mathrm{Gr} = G(\mathscr{K})/G(\mathscr{O}).$$

Here $\mathscr{K} = \mathbb{C}((t))$ and $\mathscr{O} = \mathbb{C}[[t]]$. We have endowed this set with a topology; in fact it is an ind-variety, i.e. an increasing union of complex algebraic varieties.

We have an action of $G(\mathscr{K})$ on Gr by left translation, but we are mainly interesed in the actions of various subgroups. We choose a Borel subgroup $B \subset G$, a maximal torus $T \subset B$. The *Iwahori subgroup* attached to B is the preimage of B under the morphism

$$G(\mathscr{O}) \to G$$

sending t to 0. We then consider the actions of $G(\mathscr{O})$ and I on Gr_G by left multiplication.

1.2. **Decompositions.** Let $\mathbf{X}^\vee$ be the lattice of cocharacters for T. The choice of B determines a choice of positive roots for G, hence also a cone $\mathbf{X}^\vee_+ \subset \mathbf{X}^\vee$ of dominant weights. We also have an order $\leq$ on $\mathbf{X}^\vee$, such that $\lambda \leq \mu$ iff $\mu - \lambda$ is a sum of positive coroots.

1.2.1. The decomposition of Gr into $G(\mathscr{O})$-orbits is given by the Cartan decomposition:

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \mathbf{X}^\vee_+} \mathrm{Gr}^\lambda \quad \text{where } \mathrm{Gr}^\lambda = G(\mathscr{O}) \cdot t^\lambda \cdot G(\mathscr{O})/G(\mathscr{O}).$$

Here we have

$$\dim(\mathrm{Gr}^\lambda) = \langle \lambda, 2\rho \rangle$$

where 2ρ is the sum of the positive roots, and

$$\overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\substack{\mu \in \mathbf{X}^\vee_+ \\ \mu \leq \lambda}} \mathrm{Gr}^\mu.$$

1.2.2. The decomposition into I-orbits is given by the Schubert decomposition:

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \mathbf{X}^\vee} \mathrm{Gr}_\lambda \quad \text{where } \mathrm{Gr}_\lambda = I \cdot t^\lambda \cdot G(\mathscr{O})/G(\mathscr{O}).$$

Here Gr_λ is isomorphic to an affine space, of dimension

$$\mathrm{Gr}_\lambda = |\langle \lambda, 2\rho \rangle| - \delta_\lambda$$

where δ_λ is the length of the shortest $w \in W$ such that $w\lambda$ is dominant. This decomposition is related to the previous one by

$$\mathrm{Gr}^\lambda = \bigsqcup_{\mu \in W(\lambda)} \mathrm{Gr}_\mu.$$

1.3. **Convolution diagram.** We will also consider

$$\mathrm{Conv} = G(\mathscr{K}) \times^{G(\mathscr{O})} \mathrm{Gr}$$

where $G(\mathscr{K}) \times^{G(\mathscr{O})} \mathrm{Gr}$ is the quotient of $G(\mathscr{K}) \times \mathrm{Gr}$ by the action of $G(\mathscr{O})$ given by $g \cdot (h, x) = (hg^{-1}, g \cdot x)$. We have an action of $G(\mathscr{O})$ on Conv induced by multiplication on the left on $G(\mathscr{K})$.

In terms of quotient stacks, we have morphisms

$$(G(\mathscr{O})\backslash\mathrm{Gr}) \times (G(\mathscr{O})\backslash\mathrm{Gr}) \xleftarrow{p} G(\mathscr{O})\backslash\mathrm{Conv} \xrightarrow{m} G(\mathscr{O})\backslash\mathrm{Gr} \tag{1.1}$$

given by

$$(G(\mathscr{O}) \cdot hG(\mathscr{O}), G(\mathscr{O}) \cdot x) \leftarrow\!\shortmid G(\mathscr{O}) \cdot [h : x] \mapsto G(\mathscr{O}) \cdot hx.$$

2. The Satake category

2.1. **Derived Satake category.** Recall that in Lectures 4 and 6 we have seen, given a complex algebraic variety X and a complex algebraic group H acting on it, how to define the derived category $D^{\mathrm{b}}(H\backslash X)$ of complexes of $\mathbb{C}$-sheaves on the quotient stack $H\backslash X$, i.e. "H-equivariant sheaves on X." Here we want to use this to define the category

$$D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr}).$$

This does not really make sense, since Gr is not a variety and $G(\mathscr{O})$ is not an algebraic group (it is not of finite type). One can make sense of this category as follows: one writes

$$\mathrm{Gr} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} X_n$$

where each X_n is an algebraic variety which is stable under the action of $G(\mathscr{O})$, and such that the action of $G(\mathscr{O})$ factors through an action of a quotient K_n of finite type such that the map $G(\mathscr{O}) \to G$ factors through K_n. We can choose these data in such a way that the map $G(\mathscr{O}) \to K_n$ factors through K_{n+1}. Then we set

$$D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr}) = \varinjlim_{n \geq 0} D^{\mathrm{b}}(K_n\backslash X_n)$$

where the (fully faithful!) functor

$$D^{\mathrm{b}}(K_n\backslash X_n) \to D^{\mathrm{b}}(K_{n+1}\backslash X_{n+1})$$

is the composition

$$D^{\mathrm{b}}(K_n\backslash X_n) \to D^{\mathrm{b}}(K_{n+1}\backslash X_n) \to D^{\mathrm{b}}(K_{n+1}\backslash X_{n+1})$$

where the second functor is pushforward under the embedding $X_n \to X_{n+1}$.

There exists a bifunctor

$$\star : D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr}) \times D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr}) \to D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr})$$

given by

$$\mathcal{F} \star \mathcal{G} = m_* p^* (\mathcal{F} \boxtimes \mathcal{G}).$$

(To make sense of this, one really wants to "approximate" the ind-variety Conv by algebraic varieties, but writing down the details is really cumbersome, and hence omitted.)

Fact 2.1. *This defines a monoidal structure on the category $D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr})$.*

Remark 2.2. One should think of $D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr})$ as a "categorical upgrade" of the vector space $\mathsf{H}_{\mathcal{G}}$ from Lecture 2, and of the monoidal product $\star$ as a "categorical upgrade" of the product on this vector space.

2.2. **The abelian Satake category.** In Lectures 7 and 9 we have seen how to define the "perverse t-structure" on a category $D^{\mathrm{b}}(H\backslash X)$. Using "finite-dimensional approximations" like in §2.1, this defines a perverse t-structure on $D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr})$. The heart of this t-structure is the category

$$\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$$

of $G(\mathscr{O})$-equivariant perverse sheaves on Gr.

We can now state the *geometric Satake equivalence* as follows. Consider the complex reductive group $G^\vee$ which is Langlands dual to G.

Theorem 2.3. *(1) If $\mathcal{F}, \mathcal{G}$ belong to $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$, then $\mathcal{F} \star \mathcal{G}$ belongs to the subcategory $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$. As a consequence, the bifunctor $\star$ restricts to a monoidal product on the abelian category $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$.*

(2) There exists an equivalence of monoidal categories

$$\mathsf{S} : (\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}), \star) \cong (\mathrm{Rep}(G^\vee), \otimes)$$

such that the following diagram commutes:

$$\begin{array}{ccc} (\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}), \star) & \xrightarrow[\sim]{\mathsf{S}} & (\mathrm{Rep}(G^\vee), \otimes) \\ & {\scriptstyle \mathsf{H}^\bullet(\mathrm{Gr}, -)} \searrow \quad \swarrow {\scriptstyle \mathrm{For}} & \\ & \mathrm{Vect}_{\mathbb{C}}. & \end{array}$$

2.3. **The case of a torus.** Assume that $G = T$ is a torus. In this case we have seen that Gr is discrete; in fact we have

$$\mathrm{Gr} = \mathbf{X}^\vee.$$

Since $T(\mathscr{K})$ is abelian, the action of $T(\mathscr{O})$ on Gr is trivial; as a consequence we have

$$\mathrm{Conv} = (T(\mathscr{K})/T(\mathscr{O})) \times \mathrm{Gr} = \mathbf{X}^\vee \times \mathbf{X}^\vee.$$

The convolution diagram (1.1) identifies with the diagram

$$(T(\mathscr{O})\backslash\mathbf{X}^\vee) \times (T(\mathscr{O})\backslash\mathbf{X}^\vee) \leftarrow T(\mathscr{O})\backslash(\mathbf{X}^\vee \times \mathbf{X}^\vee) \to T(\mathscr{O})\backslash\mathbf{X}^\vee$$

where the left arrow is the obvious morphism and the right one is induced by the sum morphism

$$\mathbf{X}^\vee \times \mathbf{X}^\vee \to \mathbf{X}^\vee.$$

The category $\mathrm{Perv}_{T(\mathscr{O})}(\mathrm{Gr})$ identifies with the category of finite-dimensional $\mathbf{X}^\vee$-graded vector spaces, where the component in degree $\lambda \in \mathbf{X}^\vee$ records the restriction of the perverse sheaf to the point $\lambda \in \mathbf{X}^\vee$. Under this identification, the convolution product $\star$ corresponds to the tensor product of graded vector spaces. Note that the dual group $T^\vee$ has lattice of characters $\mathbf{X}^\vee$, so that this category identifies with $\mathrm{Rep}(T^\vee)$.

2.4. **Simple objects.** We come back to the general setting.

The general theory of perverse sheaves provides a bijection between isomorphism classes of simple objects in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ and the set of pairs $(X, \mathcal{L})$ where $X \subset \mathrm{Gr}$ is a $G(\mathscr{O})$-orbit and $\mathcal{L}$ is an isomorphism class of simple $G(\mathscr{O})$-equivariant local systems on X. By the Cartan decomposition, X is of the form Gr^{λ} with $\lambda \in \mathbf{X}_+^{\vee}$. One can check that these orbits are simply connected; therefore any local system is constant, hence there is only one possible choice for $\mathcal{L}$: the rank-1 constant local system $\underline{\mathbb{C}}_{\mathrm{Gr}^{\lambda}}$.

All in all, we therefore have a bijection between the set of isomorphism classes of simple objects in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ and $\mathbf{X}_+^{\vee}$. We will denote by IC_{λ} the simple object associated with λ.

With this notation we can now state one of the main results of [2], which was mentioned in Lecture 2.

Theorem 2.4 (Lusztig)**.** *For any* $\lambda \in \mathbf{X}_+^{\vee}$ *we have*

$$M_{\lambda} = \sum_{\mu \in \mathbf{X}_+^{\vee}} \sum_{n \in \mathbb{Z}} q^{n/2} \cdot \mathrm{rk}(\mathcal{H}^n(\mathrm{IC}_{\lambda | \mathrm{Gr}^{\mu}})).$$

Proposition 2.5. *The category* $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ *is semisimple.*

Idea of proof. We consider the category $\mathrm{Perv}_{(G(\mathscr{O}))}(\mathrm{Gr})$ of perverse sheaves on Gr which are constructible with respect to the stratification by $G(\mathscr{O})$-orbits. Then we have a forgetful functor

$$\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}) \to \mathrm{Perv}_{(G(\mathscr{O}))}(\mathrm{Gr}),$$

which is fully faithful by the general theory of perverse sheaves. It therefore suffices to prove that the category $\mathrm{Perv}_{(G(\mathscr{O}))}(\mathrm{Gr})$ is semisimple. (In passing, this will prove that the functor above is an equivalence of categories.) The essential ingredient in the proof of the latter fact is the property that if $\langle \lambda, 2\rho \rangle$ is even, resp. odd, then for any $\mu \in \mathbf{X}_+^{\vee}$ such that $\mathrm{IC}_{\lambda | \mathrm{Gr}^{\mu}}$ is nonzero, this complex has nonzero cohomology objects only in even, resp. odd, degrees. □

Since the category $\mathrm{Rep}(G^{\vee})$ is also semisimple with isomorphism classes parametrized by $\mathbf{X}_+^{\vee}$ (see Lecture 1), we therefore have an equivalence of abelian categories

$$\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}) \cong \mathrm{Rep}(G^{\vee}).$$

But Theorem 2.3 is stronger: it says that this equivalence identifies convolution with tensor product, and also describes the underlying vector space of the representation corresponding to a given perverse sheaf.

References

[1] W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, 23–74, Astérisque 101–102, Soc. Math. France, 1983.

[2] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, 208–229, Astérisque **101–102**, Soc. Math. France, Paris, 1983.