LECTURE 11: GEOMETRIC SATAKE EQUIVALENCE, II

Recall that we have fixed a complex connected reductive algebraic group G, with a Borel subgroup B and a maximal torus $T \subset B$. We have the loop group $G(\mathscr{K})$, the arc group $G(\mathscr{O})$, the affine Grassmannian $\mathrm{Gr} = G(\mathscr{K})/G(\mathscr{O})$, and the "Satake category"

$$\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$$

of $G(\mathscr{O})$-equivariant perverse sheaves on Gr. Recall also that we have the (associative) convolution product $\star$ on the derived category $D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr})$ of sheaves on the quotient stack $G(\mathscr{O})\backslash\mathrm{Gr}$.

With this notation, the statement of the geometric Satake equivalence is as follows.

Theorem 0.1. *(1) If $\mathcal{F}, \mathcal{G}$ belong to $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$, then $\mathcal{F} \star \mathcal{G}$ belongs to the subcategory $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$. As a consequence, the bifunctor $\star$ restricts to a monoidal product on the abelian category $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$.*

(2) There exists an equivalence of monoidal categories

$$\mathsf{S} : (\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}), \star) \cong (\mathrm{Rep}(G^\vee), \otimes)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
(\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}), \star) & \xrightarrow[\sim]{\mathsf{S}} & (\mathrm{Rep}(G^\vee), \otimes) \\
& {\scriptstyle \mathsf{H}^\bullet(\mathrm{Gr}, -)} \searrow \quad \swarrow {\scriptstyle \mathrm{For}} & \\
& \mathsf{Vect}_{\mathbb{C}}. &
\end{array}$$

In this lecture we want to discuss the proof of this theorem.

1. Weight functors

1.1. **Semiinfinite orbits.** Consider the unipotent radical U of B, and the unipotent radical U^- of the opposite Borel subgroup. The decompositions of Gr into $U(\mathscr{K})$ and $U^-(\mathscr{K})$ are given by the *Iwasawa decompositions*:

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \mathbf{X}^\vee} S^+_\lambda \quad \text{where } S^+_\lambda = U(\mathscr{K}) \cdot t^\lambda \cdot G(\mathscr{O})/G(\mathscr{O})$$

and

$$\mathrm{Gr} = \bigsqcup_{\lambda \in \mathbf{X}^\vee} S^-_\lambda \quad \text{where } S^-_\lambda = U^-(\mathscr{K}) \cdot t^\lambda \cdot G(\mathscr{O})/G(\mathscr{O}).$$

Here each $S^\pm_\lambda$ is infinite-dimensional, but we control the dimension of their intersections with the $G(\mathscr{O})$-orbits: we have

$$\dim(\mathrm{Gr}^\lambda \cap S^+_\mu) = \langle \rho, \lambda + \mu \rangle \quad \text{and} \quad \dim(\mathrm{Gr}^\lambda \cap S^-_\mu) = \langle \rho, \lambda - \mu \rangle. \tag{1.1}$$

We also know when these intersections are nonempty: this happens precisely when μ belongs to the intersection of the convex hull of $W(\lambda)$ with $\lambda + \mathbb{Z}\mathfrak{R}$. (Note that these facts are far from trivial in general, and crucial for the whole theory!)

Finally we have

$$\overline{S^+_\lambda} = \bigsqcup_{\substack{\mu\in\mathbf{X}^\vee \\ \mu\le\lambda}} S^+_\mu$$

and

$$\overline{S^-_\lambda} = \bigsqcup_{\substack{\mu\in\mathbf{X}^\vee \\ \mu\ge\lambda}} S^-_\mu .$$

1.2. **Weight functors.** We denote by

$$s^+_\lambda : S^+_\lambda \to \mathrm{Gr}, \quad s^-_\lambda : S^-_\lambda \to \mathrm{Gr}$$

the embeddings, and one considers the functors

$$\mathsf{F}^\pm_\lambda : \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}) \to \mathsf{Vect}_\mathbb{C}$$

defined by

$$\mathsf{F}^+_\lambda(\mathcal{F}) = \mathsf{H}^\bullet_c(S^+_\lambda, (s^+_\lambda)^*\mathcal{F}), \qquad \mathsf{F}^-_\lambda(\mathcal{F}) = \mathsf{H}^\bullet(S^-_\lambda, (s^-_\lambda)^!\mathcal{F}).$$

The following property is an easy consequence of (1.1) and basic properties of cohomology.

Lemma 1.1. *For any $\mathcal{F} \in \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ we have*

$$\mathsf{H}^n_c(S^+_\lambda, (s^+_\lambda)^*\mathcal{F}) = 0 \quad \text{if } n > \langle\lambda, 2\rho\rangle,$$
$$\mathsf{H}^n(S^-_\lambda, (s^-_\lambda)^!\mathcal{F}) = 0 \quad \text{if } n < \langle\lambda, 2\rho\rangle.$$

The following theorem is more difficult. It is an application of *Braden's hyperbolic localization theorem*, see [1, Theorem 2.10.7].

Theorem 1.2. *For any $\mathcal{F} \in \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$, $\lambda \in \mathbf{X}^\vee$ and n in $\mathbb{Z}$, there exists a canonical isomorphism*

$$\mathsf{H}^n_c(S^+_\lambda, (s^+_\lambda)^*\mathcal{F}) \cong \mathsf{H}^n(S^-_\lambda, (s^-_\lambda)^!\mathcal{F}).$$

This theorem and Lemma 1.1 imply that the spaces in the theorem vanish unless $n = \langle\lambda, 2\rho\rangle$, i.e. we have

$$\mathsf{F}^+_\lambda(\mathcal{F}) = \mathsf{H}^{\langle\lambda,2\rho\rangle}_c(S^+_\lambda, (s^+_\lambda)^*\mathcal{F}), \qquad \mathsf{F}^-_\lambda(\mathcal{F}) = \mathsf{H}^{\langle\lambda,2\rho\rangle}(S^-_\lambda, (s^-_\lambda)^!\mathcal{F}).$$

Moreover these spaces are canonically isomorphic; they will therefore be denoted $\mathsf{F}_\lambda(\mathcal{F})$.

One next obtains information about the value of the functor F_λ on simple objects.

Proposition 1.3. *If $\lambda \in \mathbf{X}^\vee$ and $\mu \in \mathbf{X}^\vee_+$, the vector space $\mathsf{F}_\lambda(\mathrm{IC}_\mu)$ has a basis parametrized by the irreducible components of the intersection $S^+_\lambda \cap \mathrm{Gr}^\mu$.*

The main ingredients of the proof of this proposition are the following:

- by semisimplicity, IC_μ identifies with the perverse degree-0 cohomology of the !-pushforward of $\underline{\mathbb{C}}_{\mathrm{Gr}^\mu}[\langle 2\rho, \mu\rangle]$ along the embedding $\mathrm{Gr}^\mu \hookrightarrow \mathrm{Gr}$;
- for a variety X of degree d, $\mathsf{H}^{2d}_c(X)$ admits a basis parametrized by irreducible components of X.

An additional important property of the weight functors is the following.

Proposition 1.4. *For any $\mathcal{F} \in \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$, we have a canonical isomorphism*

$$\mathsf{H}^\bullet(\mathrm{Gr}, \mathcal{F}) = \bigoplus_{\lambda\in\mathbf{X}^\vee} \mathsf{F}_\lambda(\mathcal{F}).$$

2. Proof of the geometric Satake equivalence

2.1. **Exactness.** First we consider Item (1) in Theorem 0.1. Recall the convolution diagram

$$(G(\mathscr{O})\backslash \mathrm{Gr}) \times (G(\mathscr{O})\backslash \mathrm{Gr}) \xleftarrow{p} G(\mathscr{O})\backslash \mathrm{Conv} \xrightarrow{m} G(\mathscr{O})\backslash \mathrm{Gr}.$$

If $\mathcal{F}$ and $\mathcal{G}$ belong to $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$, it is not difficult to check that $p^*(\mathcal{F} \boxtimes \mathcal{G})$ is a perverse sheaf, and that it is constructible with respect to the stratification

$$\mathrm{Conv} = \bigsqcup_{\lambda,\mu \in \mathbf{X}_+^\vee} \pi^{-1}(\mathrm{Gr}^\lambda) \times^{G(\mathscr{O})} \mathrm{Gr}^\mu. \tag{2.1}$$

(Here $\pi : G(\mathscr{K}) \to \mathrm{Gr}$ is the natural quotient morphism, and we mean that the restriction of each cohomology complex to each stratum is a constant sheaf.) The exactness statement therefore follows from the fact that the functor

$$m_* : D^{\mathrm{b}}(G(\mathscr{O})\backslash \mathrm{Conv}) \to D^{\mathrm{b}}(G(\mathscr{O})\backslash \mathrm{Gr})$$

sends perverse sheaves constructible with respect to the stratification (2.1) to perverse sheaves. This property follows from an analysis of the dimension of intersections of fibers of m with strata; this is formalized in the notion of "stratified semismall map." (For details on this notion, see [1, Definition 3.8.8]. For a proof that m is stratified semismall, see [1, Proposition 9.5.4]. The proof is based on the formulas (1.1).)

Remark 2.1. There is another known proof of Theorem 0.1(1), based on Lusztig's results mentioned in Lectures 2 and 9. For details, see [1, Exercise 9.4.1] or [2, Proposition 2.2.1].

2.2. **Tannakian formalism.** As the next step, one checks that the functor

$$\mathsf{H}^\bullet(\mathrm{Gr}, -) : \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}) \to \mathsf{Vect}_{\mathbb{C}}$$

is monoidal. One can also check that it is faithful. (This follows from Propositions 1.3 and 1.4, which show that the functor $\mathsf{H}^\bullet(\mathrm{Gr}, -)$ does not send any nonzero object to 0.) Tannakian formalism therefore provides for us a Hopf algebra C_G and an equivalence of monoidal categories

$$\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}) \xrightarrow{\sim} \mathsf{Comod}(C_G)$$

(where $\mathsf{Comod}(C_G)$ is the category of finite-dimensional comodules over C_G) such that the forgetful functor

$$\mathsf{Comod}(C_G) \to \mathsf{Vect}_{\mathbb{C}}$$

corresponds to

$$\mathsf{H}^\bullet(\mathrm{Gr}, -) : \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}) \to \mathsf{Vect}_{\mathbb{C}}.$$

The rest of the proof will consist in analyzing the Hopf algebra C_G, and finally identifying it with the algebra of functions on the dual group $G^\vee$.

2.3. **Commutativity.** The first property one should check is that the product on C_G is commutative. By Tannakian formalism, this property is equivalent to the existence of a "commutativity constraint" for the convolution product $\star$, i.e. a bifunctorial isomorphism

$$\mathcal{F} \star \mathcal{G} \cong \mathcal{G} \star \mathcal{F}$$

for $\mathcal{F}, \mathcal{G}$ in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ compatible with the other structures we have on $\star$. This construction is far from obvious, and is due to Drinfeld. It uses the "moduli" description of Gr, i.e. the fact that this ind-scheme parametrizes the following data.

We fix a smooth curve C and a point $x \in C$. Then the data are those of a G-torsor on C and a trivialization of this torsor away from x, i.e. an isomorphism between the restriction of the torsor to $C \smallsetminus \{x\}$ and the trivial torsor $G \times (C \smallsetminus \{x\})$. One can construct a similar ind-scheme over C^2, and then "move the points in C^2" to produce the commutativity constraint.

2.4. **Identification of the group scheme.** Now we know that C_G is a *commutative* Hopf algebra over $\mathbb{C}$, hence the algebra of functions on a group scheme $\mathfrak{G}$ over $\mathbb{C}$. Using Tannakian formalism one successively checks that this group scheme has the following properties:

(1) $\mathfrak{G}$ is of finite type (i.e. C_G is finitely generated as an algebra). (This uses basic facts about the simple objects in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$.)
(2) $\mathfrak{G}$ is connected. (Same.)
(3) $\mathfrak{G}$ is is reductive. (This follows from the semisimplicity of the category $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$.)

The next step is to produce a maximal torus in $\mathfrak{G}$. Recall that this maximal torus should be $T^\vee$ and that, by Tannakian formalism, producing a morphism of algebraic groups $T^\vee \to \mathfrak{G}$ is equivalent to producing a monoidal functor

$$\mathrm{Rep}(\mathfrak{G}) \to \mathrm{Rep}(T^\vee) \tag{2.2}$$

compatible with the natural functors to $\mathsf{Vect}_{\mathbb{C}}$. Now, since $\mathbf{X}^\vee$ is the character lattice of $T^\vee$, we have a canonical equivalence between $\mathrm{Rep}(T^\vee)$ and the category $\mathsf{Vect}_{\mathbb{C}}^{\mathbf{X}^\vee}$ of finite-dimensional $\mathbb{C}$-vector spaces graded by the abelian group $\mathbf{X}^\vee$. And we have a functor

$$\mathrm{Rep}(\mathfrak{G}) \to \mathsf{Vect}_{\mathbb{C}}^{\mathbf{X}^\vee}$$

sending $\mathcal{F}$ to

$$\bigoplus_{\lambda \in \mathsf{X}^\vee} \mathsf{F}_\lambda(\mathcal{F}).$$

This produces the desired functor (2.2). One checks that this functor is indeed monoidal, and Proposition 1.4 says that it is compatible with the natural functors to $\mathsf{Vect}_{\mathbb{C}}$.

This process produces a morphism $T^\vee \to \mathfrak{G}$. It is not difficult to check that this morphism is an embedding, and that its image is a maximal torus in $\mathfrak{G}$.

Finally, it remains to determine the root datum of $\mathfrak{G}$ with respect to $T^\vee$, and more precisely to identify it with $(\mathbf{X}^\vee, \mathfrak{R}^\vee, \mathbf{X}, \mathfrak{R})$. The essential ingredients for that are:

- the fact that the simple $\mathfrak{G}$-modules are parametrized by $\mathbf{X}^\vee_+$ (because we know the classification of simple objects in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$);
- Proposition 1.3, together with the criterion for the intersection $S^+_\lambda \cap \mathrm{Gr}^\mu$ to be nonempty, which determines the $T^\vee$-weights of the simple $\mathfrak{G}$-modules.

References

[1] P. Achar, *Perverse sheaves and applications to representation theory*, Math. Surveys Monogr. 258, American Mathematical Society, Providence, RI, 2021.

[2] V. Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, preprint arXiv:alg-geom/9511007.