K-theory of Steinberg variety

I. Grothendieck groups

$\mathcal{A}$ - an abelian category

$K_0(\mathcal{A})$: free ab gp on obj in $\mathcal{A}$

relation: $0 \to A \to B \to C \to 0 \rightsquigarrow [B] = [A] + [C]$.

Generalize: given a filtr.

$0 \subset B_1 \subset \cdots \subset B_k = B$ in $\mathcal{A}$

then $[B] = [B_1] + [B_2/B_1] + \cdots + [B_k/B_{k-1}]$.

in $K(\mathcal{A})$.

Examples.

1) $\mathcal{A}$ = fin dim'l vec sp / $\mathbb{R}$.

$[\mathbb{R}^n] = n \cdot [\mathbb{R}]$.

$K(\mathcal{A}) = \mathbb{Z}$.

2) $\mathcal{A}$ = arb. vec sp / $\mathbb{R}$

$K(\mathcal{A}) = 0$ (why?)

3) $\mathcal{A}$ = fin dim reps of a group.

$K(\mathcal{A}) = \mathbb{Z}[\mathrm{Irr}(G)]$.

II. K-theory of a variety

X - a variety, G a group

$G \circlearrowright X$. Look at $\mathrm{Coh}^G(X)$ abelian cat.

$K^G(X) := K(\mathrm{Coh}^G(X))$.

Main examples:

1) X affine, $= \mathrm{Spec}\ \underbrace{\mathbb{C}[x_1, \ldots, x_n]/I}_{R:=}$.

Then $\mathrm{Coh}^G(X) =$ fin. gen. R-modules w/ compat G-action.

2) $X = G/H$. Then

$\mathrm{Coh}^G(X) = \mathrm{Coh}^H(pt) = \mathrm{Rep}(H)$.

III. Some K-theory calc.

$G = GL_n$ or conn. reduc. gp

$B = \begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{bmatrix}$ or Borel subgp

$T = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix}$ or max torus.

$\mathbb{X} = \mathbb{Z}^n$ or characters of T.

$\mathrm{Irr}(T) = \mathbb{X} = \mathrm{Irr}(B)$

Because $B = T \ltimes \begin{bmatrix} 1 & \cdots & * \\ & \ddots & \\ & & 1 \end{bmatrix}$ acts trivially on any irrep by Engel's Thm.

1) $\mathrm{Coh}^B(pt) = \mathrm{Rep}(B)$. ring

$K^B(pt) = \mathbb{Z}[\mathrm{Irr}(B)] = \mathbb{Z}[\mathbb{X}]$.

2) $\mathrm{Coh}^G(G/B) \simeq \mathrm{Coh}^B(pt)$

flag variety $\{F_\bullet = F_0 \subset F_1 \subset \cdots \subset F_n\}$

$K^G(G/B) = \mathbb{Z}[\mathbb{X}]$.

3) "Springer resol'n"

$\tilde{\mathcal{N}} = T^*G/B = G \times^B (\mathfrak{g}/\mathfrak{b})^*$

via Killing form (not quite):

$(\mathfrak{g}/\mathfrak{b})^* \simeq \mathrm{Lie}(U)$

$\tilde{\mathcal{N}} = \{(gF_\bullet^{std}, \underset{\mathfrak{g}}{\overset{x}{\cap}}) \mid \mathrm{Ad}(g)^{-1}x \in \mathrm{Lie}$

$= \{(F_\bullet, x) \mid x$ nilpotent, preserves $F_\bullet\}$

$K^G(\tilde{\mathcal{N}}) = K^B(\mathrm{Lie}(u))$

$= K$(B-equt modules / poly ring)

$= \mathbb{Z}[\mathbb{X}]$ again. Idea: Hilb. basis Thm.

3') Let $G \times \mathbb{C}^\times \curvearrowright \tilde{\mathcal{N}} = T^*G/B$ by: $\mathbb{C}^\times$ acts on $\mathrm{Lie}(U)$ with weight 2.

$$K^{G\times\mathbb{C}^\times}(\tilde{\mathcal{N}}) = \mathbb{Z}[\mathrm{Irr}(T\times\mathbb{C}^\times)] = \mathbb{Z}[v,v^{-1}][\mathbb{X}].$$

4) Steinberg variety

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(F_\bullet, F_\bullet', x) \mid x \text{ nilpotent, preserves 2 flags}\}.$$

Most complicated so far.

Cut into pieces: $w \in W$,

$$Z_w = \{(F_\bullet, F_\bullet', x) \mid F_\bullet, F_\bullet' \text{ in rel. position } w\}.$$

Fact: $Z_w = G \times^{B \cap wBw^{-1}} (\mathrm{Lie}(U \cap wUw^{-1}))$

$$K^{G\times\mathbb{C}^\times}(Z_w) = \mathbb{Z}[v,v^{-1}][\mathbb{X}].$$

<u>Thm.</u> $K^{G\times\mathbb{C}^\times}(Z)$ contains $K^{G\times\mathbb{C}^\times}(Z_e) = \mathbb{Z}[v,v^{-1}][\mathbb{X}]$ as a subring.

- $K^{G\times\mathbb{C}^\times}(Z)$ is free over $\mathbb{Z}[v,v^{-1}][\mathbb{X}]$ of rank $|W|$.

What ring is it?

<u>IV. Bernstein presentation</u>

Recall:

H_{aff} = "generic" version of functions on $I\backslash G_F/I$.

- algebra over $\mathbb{Z}[v,v^{-1}]$.

$$H_{aff}|_{v=1} = \mathbb{Z}[W_{aff}].$$

H_{fin} = "generic" version of functions on $B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)$.

$$H_{fin}|_{v=1} = \mathbb{Z}[W].$$

<u>Thm</u> (Bernstein)

$$H_{aff} = H_{fin} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Z}[v,v^{-1}][\mathbb{X}]$$

as modules, <u>not</u> rings.

Ring structure: "semidirect product".

<u>Thm</u> (Kazhdan-Lusztig)

$$K^{G\times\mathbb{C}^\times}(Z) \cong H_{aff}$$

as rings.