LECTURE 14: GAITSGORY'S CENTRAL FUNCTOR AND WAKIMOTO SHEAVES

Recall from Lectures 2 and 13 the spherical affine Hecke algebra

$$\mathscr{C}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$$

and the affine Hecke algebra

$$\mathscr{C}(\mathcal{I}\backslash\mathcal{G}/\mathcal{I}).$$

(Here $\mathcal{K} \subset \mathcal{G}$ is the standard maximal open compact subgroup, and $\mathcal{I} \subset \mathcal{K}$ is the Iwahori subgroup determined by our choice of Borel subgroup.) A theorem of Bernstein asserts that $\mathscr{C}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$ identifies canonically with the center of $\mathscr{C}(\mathcal{I}\backslash\mathcal{G}/\mathcal{I})$. More specifically, consider the space

$$\mathscr{C}(\mathcal{K}\backslash\mathcal{G}/\mathcal{I}).$$

of $(\mathcal{K},\mathcal{I})$-invariant locally constant functors on $\mathcal{G}$ with compact support, and the map

$$\pi : \mathscr{C}(\mathcal{I}\backslash\mathcal{G}/\mathcal{I}) \to \mathscr{C}(\mathcal{K}\backslash\mathcal{G}/\mathcal{I})$$

defined by

$$\pi(f)(g) = \int_{\mathcal{K}} f(x \cdot g)\mathrm{d}x.$$

It is easily seen that this map restricts to an algebra map

$$Z(\mathscr{C}(\mathcal{I}\backslash\mathcal{G}/\mathcal{I})) \to \mathscr{C}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K}),$$

and Bernstein's theorem states that the latter map is an isomorphism.

Gaitsgory's central functor is a geometric counterpart of the inverse to this isomorphism.

1. The central functor

1.1. **Nearby cycles.** We start by explaining a general construction for sheaves. Consider a complex algebraic variety X, and an algebraic functor $f : X \to \mathbb{C}$. We set

$$X^\times := f^{-1}(\mathbb{C}^\times), \quad X^0 := f^{-1}(0),$$

and consider the diagram

$$\begin{array}{ccccccc}
\widetilde{X} & \xrightarrow{\exp_X} & X^\times & \xrightarrow{j} & X & \xleftarrow{i} & X^0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
\mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times & \longrightarrow & \mathbb{C} & \longleftarrow & \{0\}
\end{array}$$

where all the squares are cartesian and the unlabelled arrow on the bottom line are the obvious embeddings. The *nearby cycles functor* associated with f is the functor

$$\Psi_f : D^+(X^\times) \to D^+(X^0)$$

defined by

$$\Psi_f(\mathscr{F}) = i^* j_* (\exp_X)_* (\exp_X)^* \mathscr{F}[-1].$$

The action of $\mathbb{Z}$ on $\mathbb{C}$ over $\mathbb{C}^\times$ (where 1 acts by $z \mapsto z + 2i\pi$) induces a canonical automorphism of the functor Ψ_f, called the monodromy automorphism.

The basic properties of nearby cycles are as follows:

(1) The functor Ψ_f sends bounded constructible complexes to bounded constructible complexes (cf. [1, Theorem 4.2.3]).
(2) The functor Ψ_f sends perverse sheaves to perverse sheaves (cf. [1, Theorem 4.2.8]).
(3) Nearby cycles commute with smooth pullback in the sense that given a smooth map $\phi : Y \to X$, for any $\mathscr{F}$ in $D^+(X^\times)$ there exists a canonical isomorphism
$$\Psi_{f\circ\phi}((\phi^\times)^*\mathscr{F}) \cong (\phi^0)^*\Psi_f(\mathscr{F}),$$
see [1, Lemma 4.1.7(2)]. (Here $\phi^\times$ and ϕ^0 are the restrictions of ϕ to $Y^\times = (f\circ\phi)^{-1}(\mathbb{C}^\times)$ and $Y^0 = (f\circ\phi)^{-1}(0)$ respectively.)
(4) Nearby cycles commute with proper pushforward in the sense that given a proper map $\phi : Y \to X$, for any $\mathscr{F}$ in $D^+(Y^\times)$ there exists a canonical isomorphism
$$\Psi_f((\phi^\times)_*\mathscr{F}) \cong (\phi^0)_*\Psi_{f\circ\phi}(\mathscr{F}),$$
see [1, Lemma 4.1.7(1)].

1.2. **Construction of Gaitsgory's functor.** We turn to the context of the geometric Satake equivalence. We have a complex connected reductive algebraic group G, with a Borel subgroup B and a maximal torus $T \subset B$. We have the loop group $G(\mathscr{K})$, the arc group $G(\mathscr{O})$, the affine Grassmannian $\mathrm{Gr}_G = G(\mathscr{K})/G(\mathscr{O})$, and the "Satake category"
$$\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}_G),$$
which is the heart of the perverse t-structure on $D^{\mathrm{b}}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O}))$. The triangulated category $D^{\mathrm{b}}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O}))$ has a monoidal product $\star^{G(\mathscr{O})}$, which restricts to a monoidal product on $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}_G)$.

We also consider the Iwahori subgroup $I \subset G(\mathscr{O})$ (i.e. the inverse image of B under the natural morphism $G(\mathscr{O}) \to G$), and the affine flag variety $\mathrm{Fl}_G = G(\mathscr{K})/I$. This is an ind-scheme, and we have a canonical morphism
$$\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G$$
which is a locally trivial fibration with fibers isomorphic to $G(\mathscr{O})/I \cong G/B$. Then we have the derived category
$$D^{\mathrm{b}}(I\backslash G(\mathscr{K})/I),$$
and the heart $\mathrm{Perv}_I(\mathrm{Fl}_G)$ of the perverse t-structure on it. Here again we have a monoidal product $\star^I$ on $D^{\mathrm{b}}(I\backslash G(\mathscr{K})/I)$, but it does *not* restrict to a monoidal product on $\mathrm{Perv}_I(\mathrm{Fl}_G)$: a product of perverse sheaves is *not* perverse in general.

Gaitsgory considers an ind-scheme $\mathrm{Gr}_{G,\mathbb{C}}$ over $\mathbb{C}$ which has the property that
$$\mathrm{Gr}_{G,\mathbb{C}|\mathbb{C}^\times} = \mathrm{Gr}_G \times \mathbb{C}^\times, \quad \mathrm{Gr}_{G,\mathbb{C}|\{0\}} = \mathrm{Fl}_G.$$
(This ind-scheme is defined in terms of the description of Gr_G as a certain moduli space of G-torsors on $\mathbb{A}^1_{\mathbb{C}}$. One should think of it as a deformation of the affine Grassmannian to the affine flag variety.) Then he defines the functor
$$\mathsf{Z} : D^{\mathrm{b}}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O})) \to D^{\mathrm{b}}(I\backslash G(\mathscr{K})/I)$$

by setting

$$\mathsf{Z}(\mathscr{F}) = \Psi(p^*\mathscr{F}[1])$$

where $p : \mathrm{Gr}_{G,\mathbb{C}|\mathbb{C}^\times} \to \mathrm{Gr}_G$ is the projection and Ψ is the nearby cycles functor associated with the structure map $\mathrm{Gr}_{G,\mathbb{C}} \to \mathbb{C}$.

The following statement gathers some of the main results of [4]. The proofs essentially boil down to applications of the properties of nearby cycles recalled in §1.1. (For details, see e.g. [2].)

Theorem 1.1 (Gaitsgory, [4]). *(1) The functor Z is monoidal (with respect to the products $\star^{G(\mathscr{O})}$ and $\star^I$) and t-exact (i.e. it sends perverse sheaves to perverse sheaves).*

(2) For any $\mathscr{F}$ in $D^{\mathrm{b}}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O}))$ and $\mathscr{G}$ in $D^{\mathrm{b}}(I\backslash G(\mathscr{K})/I)$ there exists a canonical isomorphism

$$\mathsf{Z}(\mathscr{F}) \star^I \mathscr{G} \cong \mathscr{G} \star^I \mathsf{Z}(\mathscr{F}).$$

Moreover, these objects are perverse sheaves.

(3) For any $\mathscr{F}$ in $D^{\mathrm{b}}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O}))$ there exists a canonical isomorphism $\pi_* \circ \mathsf{Z}(\mathscr{F}) \cong \mathscr{F}$.

(4) For any $\mathscr{F}$ in $D^{\mathrm{b}}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O}))$, the monodromy automorphism of $\mathsf{Z}(\mathscr{F})$ is unipotent.

2. Wakimoto sheaves

2.1. **Standard and costandard perverse sheaves.** Recall the (extended) affine Weyl group

$$W_{\mathrm{ext}} = W \ltimes X_*(T).$$

Any $\lambda \in X_*(T)$ determines an element $z^\lambda \in T(\mathscr{K}) \subset G(\mathscr{K})$, and for any $w \in W$ we choose a representative $\dot{w} \in N_G(T)$. If $x = w \ltimes \lambda \in W_{\mathrm{ext}}$, we set

$$\mathrm{Fl}_{G,x} := I\dot{w}z^\lambda I/I \quad \subset \mathrm{Fl}_G.$$

This is a locally closed subvariety of Fl_G, isomorphic to an affine space, and we denote by

$$j_w : \mathrm{Fl}_{G,w} \to \mathrm{Fl}_G$$

the embedding. We also set

$$\ell(w) = \dim(\mathrm{Fl}_{G,w}).$$

(For those who know the theory: this is the length function for a "quasi-Coxeter structure" on Fl_G.)

An affine analogue of the Bruhat decomposition says that

$$\mathrm{Fl}_G = \bigsqcup_{w \in W_{\mathrm{ext}}} \mathrm{Fl}_{G,w}.$$

For $w \in W_{\mathrm{ext}}$ we set

$$\Delta_w = (j_w)_! \underline{\mathbb{C}}_{\mathrm{Fl}_{G,w}}[\ell(w)], \quad \nabla_w = (j_w)_* \underline{\mathbb{C}}_{\mathrm{Fl}_{G,w}}[\ell(w)].$$

These are *perverse sheaves*; they are called the standard and costandard perverse sheaf associated with w respectively. (This is not completely obvious, and follows from the fact that j_w is an *affine morphism*.) There exists (up to scalar) a unique nonzero morphism $\Delta_w \to \nabla_w$. Its image is a simple perverse sheaf, denoted IC_w, and the objects $(\mathrm{IC}_w : w \in W_{\mathrm{ext}})$ are representatives for isomorphism classes of simple objects in $\mathrm{Perv}_I(\mathrm{Fl}_G)$.

Note that for $w = e$ (neutral element in W_{ext}) we have $\Delta_e = \nabla_e = \mathrm{IC}_e$, and this object is the unit object for the product $\star^I$.

The following statement gathers the main properties of the standard and costandard objects with respect to convolution. (For proofs, see [2, §4.1.2–4.1.3].)

Proposition 2.1. *(1) For any $w, y \in W_{\mathrm{ext}}$ such that $\ell(wy) = \ell(w) + \ell(y)$, there exist canonical isomorphisms*

$$\Delta_w \star^I \Delta_y \cong \Delta_{wy}, \quad \nabla_w \star^I \nabla_y \cong \nabla_{wy}.$$

(2) For any $w \in W_{\mathrm{ext}}$ there exist canonical isomorphisms

$$\Delta_w \star^I \nabla_{w^{-1}} \cong \Delta_e \cong \nabla_{w^{-1}} \star^I \Delta_w.$$

(3) For any $w, y \in W_{\mathrm{ext}}$, the object $\Delta_w \star^I \nabla_y$ is perverse.

2.2. **Wakimoto perverse sheaves.** For $\lambda \in X_*(T)$, choose $\lambda_1, \lambda_2 \in X_*(T)^+$ such that $\lambda = \lambda_1 - \lambda_2$, and set

$$\mathscr{W}_\lambda = \nabla_{\lambda_1} \star^I \Delta_{-\lambda_2}.$$

It turns out that this object does not depend on the choice of λ_1, λ_2. (This follows from the fact that for $\lambda \in X_*(T)^+$ we have $\ell(\lambda) = \langle \lambda, 2\rho \rangle$; in particular, for $\lambda, \mu \in X_*(T)^+$ we have $\ell(\lambda + \mu) = \ell(\lambda) + \ell(\mu)$.)

Proposition 2.2 (Mirković)**.** *(1) For any $\lambda \in X_*(T)$ the object $\mathscr{W}_\lambda$ is a perverse sheaf, and it is supported on $\overline{\mathrm{Fl}_{G,\lambda}}$.*

(2) For any $\lambda, \mu \in X_*(T)$ we have

$$\mathscr{W}_\lambda \star^I \mathscr{W}_\mu \cong \mathscr{W}_{\lambda+\mu}.$$

For proofs, see [2, §4.2].

2.3. **Wakimoto filtrations of central sheaves.** We will say that an object $\mathscr{F} \in \mathrm{Perv}_I(\mathrm{Fl}_G)$ *admits a Wakimoto filtration* if it admits a finite filtration

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_{n-1} \subset \mathscr{F}_n = \mathscr{F}$$

such that each $\mathscr{F}_i/\mathscr{F}_{i-1}$ is of the form $\mathscr{W}_{\lambda_i}$ for some $\lambda_i \in X_*(T)$. Such a filtration (when it exists) might not be unique, but for any $\lambda \in X_*(T)$ the number of occurrences of $\mathscr{W}_\lambda$ as a subquotient is uniquely determined, and called the multiplicity of $\mathscr{W}_\lambda$ in $\mathscr{F}$. (For details, see [2, §4.3].)

Theorem 2.3 (Arkhipov–Bezrukavnikov, [3])**.** *(1) For any $\mathscr{F}$ in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}_G)$, the perverse sheaf $\mathsf{Z}(\mathscr{F})$ admits a Wakimoto filtration.*

(2) For any $\lambda \in X_*(T)$, the multiplicity of $\mathscr{W}_\lambda$ in $\mathsf{Z}(\mathscr{F})$ is the dimension of the λ-weight space of $\mathsf{Sat}(\mathscr{F})$.

This proof is rather formal. For the 1st point, one only uses the fact that $\mathsf{Z}(\mathscr{F})$ is central and convolution-exact (cf. Theorem 1.1(2)); see [2, §4.4.2] for details. For the second point, one just needs to compute the cohomology of Wakimoto sheaves with support in $U^-(\mathscr{K})$-orbits in Fl_G; see [2, Proposition 4.6.5].

References

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[3] S. Arkhipov and R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group* (with an appendix by R. Bezrukavnikov and I. Mirković), Israel J. Math. **170** (2009), 135–183.

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