Iwahori-Whittaker & Arkhipov-Bezrukavnikov ①

I. Whittaker sheaves

Work over $\overline{\mathbb{F}}_p$, not $\mathbb{C}$.

$\mathbb{A}^1$ is NOT simply conn:

$$x^p - x : \mathbb{A}^1 \to \mathbb{A}^1$$

is
- surj.
- proper (in fact, finite)
- "local" homeo. because derivative is always $\neq 0$.

This is a Galois cov., Gal gp $= \mathbb{Z}/p$.

⇝ Get a loc sys $\underline{AS}$ (of "$\mathbb{C}$" vec sp) on which Gal gp acts by p^{th} roots of 1.

Look at $U = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \subset \mathbb{P}^1$.

Orbits: 

A sheaf ~~F~~ $\mathcal{F}$ on $\mathbb{P}^1$ is:

- equivariant if: $U \times \mathbb{P}^1 \underset{pr}{\overset{a}{\rightrightarrows}} \mathbb{P}^1$

  Recall $a^*\mathcal{F} = pr^*\mathcal{F}$, etc. $= \underline{\mathbb{C}} \boxtimes \mathcal{F}$. $\boxed{\text{2 IC's, other obj.}}$

- Whitt.-equivariant if $a^*\mathcal{F} = \underline{AS} \boxtimes \mathcal{F}$.
  - NOT possible if $\mathcal{F}$ supported on 0-dim'l orbit.
  - $j_!\underline{AS} = j_*\underline{AS}$.
  - $D^b(\text{Whitt. eq. on } \mathbb{P}^1) \simeq D^b \mathrm{Vect}_{\mathbb{C}}$.

~~Fact~~. Exercise. "Whittaker-averaging on $\mathbb{P}^1$".

II. Iwahori-Whittaker sheaves

Work on $Fl_G = G(\mathcal{K})/I$.

Also: I^- = opposite Iwahori.

$$I^- = T \ltimes I^-_u$$

torus ← ∞-dim'l unipotent.

Take "generic" $\chi : I^-_u \to \mathbb{A}^1$ homom.

~~Look at~~ Goal: Say something about twisted I^-_u-equiv. w.r.t. $\chi^*\underline{AS}$.

Generalities:

I-orbits on Fl ⟷ W_{ext}

I^--orbits on Fl ⟷ $W \ltimes X_*(T)$ $(= W_{ext})$

$= I^-_u$-orbits on Fl.

Lemma. ~~Fix~~ Let $w \in W_{ext}$. TFAE:

1) $I^-_u \dot{w} I/I$ admits a $\chi^*\underline{AS}$-tw. equiv. loc. sys.
2) $I^-_u \dot{w} I/I$ is dense in $G(\mathcal{O})\dot{w}I/I$ ($\simeq$ aff. bdle over G/B)
3) w is minimal in the coset Ww.

Notation:

$$\mathrm{Perv}_{IW}(Fl) = \mathrm{Perv}_{I^-_u, \chi^*\underline{AS}}(Fl).$$

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Irr. obj in $\mathrm{Perv}_{IW}(Fl)$
$\longleftrightarrow W\backslash W_{ext} \longleftrightarrow X_*(T)$.

$$K(\mathrm{Perv}_{IW}(Fl)) \cong \mathbb{Z}[W] = K(\mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}}))$$

<u>Thm</u> [Arkhipov - Bezrukavnikov]
$\exists$ equiv. of tri. cat

$$D^b\mathrm{Perv}_{IW}(Fl) \cong D^b\mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}})$$

$$\overset{\shortparallel}{D^b_{IW}(Fl)}. \qquad \overset{\shortparallel}{D^b\mathrm{Coh}^{\check{B}}(\check{\mathfrak{u}})}$$

<u>Plan</u> ~~First~~ Look at "easiest" obj on coherent side

- free sheaves $V \otimes \mathcal{O}(\check{\mathfrak{u}})$ or $\mathcal{O}(\tilde{\mathcal{N}})$
- (<u>dominant</u> or <u>all</u>) line bdles $\mathbb{C}_\lambda \otimes \mathcal{O}(\check{\mathfrak{u}})$.

- Predict where they should go on perverse side
- Use prediction to <u>define</u> a <u>functor</u>. This is the hardest part.

Once the functor is defined, checking that it's an equiv. is not bad.

III. Carrying out the plan

Key tool: "Whittaker averaging"

$$\mathrm{Av}_{IW}: D^b_I(Fl) \to D^b_{IW}(Fl).$$

p'-version in exercise.

<u>Predictions</u>

1) $V \in \mathrm{Rep}(\check{G})$

$V \otimes \mathcal{O} \in \mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}})$

$\mathrm{Sat}(V) \in \mathrm{Perv}_{G(\mathcal{O})}(Gr)$

$Z(V) \in \mathrm{Perv}_I(Fl)$

$\mathrm{Av}_{IW} Z(V) \in \mathrm{Perv}_{IW}$

2) $\lambda \in X_*(T) = \mathrm{Irr}(\mathrm{Rep}(\check{B}))$

$\mathbb{C}_\lambda \otimes \mathcal{O} \in \mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}})$

$\mathcal{W}_\lambda \in \mathrm{Perv}_I(Fl)$

$\mathrm{Av}_{IW} \mathcal{W}_\lambda$

2a) λ dominant: $\nabla_\lambda \in \mathrm{Perv}_I(Fl)$

$\nabla_\lambda^{IW} \in \mathrm{Perv}_{IW}(Fl)$

IV. Why Iwahori-Whittaker?

$$D^b\mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}} \times_{\check{\mathfrak{g}}} \tilde{\mathcal{N}}) \qquad D^b(I\backslash Fl)$$

$\downarrow$ project onto right factor $\qquad \downarrow$

$$D^b(\mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}})) \qquad ?$$

make $K(\mathrm{Coh}^{\check{G}}(\tilde{\mathcal{N}}))$ into a right module for $\mathbb{Z}[W_{ext}]$

should be a right module matching this

antispherical module.