LECTURE 16: THE BEZRUKAVNIKOV EQUIVALENCE

1. Statement

1.1. **Data on the constructible side.** Recall that we have fixed a complex connected reductive algebraic group G, with a Borel subgroup B and a maximal torus $T \subset B$. We have the loop group $G(\mathscr{K})$, the arc group $G(\mathscr{O})$, the Iwahori subgroup $I \subset G(\mathscr{O})$ (= inverse image of B under $G(\mathscr{O}) \to G$), the affine Grassmannian $\mathrm{Gr}_G = G(\mathscr{K})/G(\mathscr{O})$, and the affine flag variety $\mathrm{Fl} = G(\mathscr{K})/I$. We also have the pro-unipotent radical $I_u \subset I$ (= inverse image of the unipotent radical of B under $G(\mathscr{O}) \to G$), and we set

$$\widetilde{\mathrm{Fl}}_G = G(\mathscr{K})/I_u.$$

We consider the following categories:

(1) $D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr}_G)$, with the monoidal product $\star^{G(\mathscr{O})}$, which stabilizes the subcategory $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}_G)$ of perverse sheaves.
(2) $D^{\mathrm{b}}(I\backslash\mathrm{Fl}_G)$, with the monoidal product $\star^I$.
(3) $D^{\mathrm{b}}(I_u\backslash\mathrm{Fl}_G)$; this is a right module for $D^{\mathrm{b}}(I\backslash\mathrm{Fl}_G)$, for an action bifunctor $\star^I$.
(4) $D^{\mathrm{b}}_{I_u,I_u}$: full triangulated subcategory of $D^{\mathrm{b}}(I_u\backslash\widetilde{\mathrm{Fl}}_G)$ generated by object obtained by pullback from $D^{\mathrm{b}}(I_u\backslash\mathrm{Fl}_G)$. This is a monoidal category[1] for a convolution product $\star^{I_u}$; its acts on the left on $D^{\mathrm{b}}(I_u\backslash\mathrm{Fl}_G)$.

Grothendieck's "faisceaux-fonctions" dictionary predicts that the category in (2) should be considered as a categorical incarnation of the affine Hecke algebra of Lectures 13–14, and that the category in (3) is a(nother) categorification of the free rank-1 right module over this algebra. (More specifically, these categories are more closely related to the group algebra of $W_{\mathrm{ext}} = W \ltimes X_*(T)$ than to the Hecke algebra.) The category in (4) is a variant for which the geometry on the dual side will turn out to be nicer.

We also Gaitsgory's central functor

$$\mathsf{Z} : D^{\mathrm{b}}(G(\mathscr{O})\backslash\mathrm{Gr}_G) \to D^{\mathrm{b}}(I\backslash\mathrm{Fl}_G)$$

and the Wakimoto sheaves $(\mathscr{W}_\lambda : \lambda \in X_*(T))$ (see Lecture 15).

1.2. **Data on the coherent side.** We have $G^\vee$ the Langlands dual group (constructed canonically from the geometric Satake equivalence), with a Borel subgroup $B^\vee$ and a maximal torus $T^\vee \subset G^\vee$ such that $X^*(T^\vee) = X_*(T)$. The *Springer resolution* is

$$\widetilde{\mathcal{N}} = T^*(G^\vee/B^\vee) = G^\vee \times^{B^\vee} \mathfrak{u}^\vee$$

where $\mathfrak{u}^\vee$ is the Lie algebra of the unipotent radical of $B^\vee$. We have a canonical morphism

$$\widetilde{\mathcal{N}} \to \mathfrak{g}^\vee$$

[1]Here this term is used in a weaker sense: there is no monoidal unit in this category.

(where $\mathfrak{g}^\vee$ is the Lie algebra of $G^\vee$). We will also consider the *Grothendieck resolution*

$$\widetilde{\mathfrak{g}^\vee} = G^\vee \times^{B^\vee} \mathfrak{b}^\vee$$

where $\mathfrak{b}^\vee$ is the Lie algebra of $B^\vee$. We have $\widetilde{\mathcal{N}} \subset \widetilde{\mathfrak{g}^\vee}$ and a canonical morphism

$$\widetilde{\mathfrak{g}^\vee} \to \mathfrak{g}^\vee$$

which extends the similar morphism for $\widetilde{\mathcal{N}}$.

We consider three versions of the Steinberg variety:

$$\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}, \quad \widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}, \quad \widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}.$$

(In the first case we need to consider a *derived fiber product*, which is to fiber products what derived tensor products are to tensor products. This a "derived scheme" in an appropriate sense. For the other versions, it turns out that the derived fiber products coincide with the ordinary fiber products, so that this subtlety can be ignored.) These (derived) schemes have canonical (diagonal) actions of $G^\vee$, so that we can consider the corresponding derived categories of equivariant coherent sheaves

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}), \quad D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}), \quad D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}).$$

Instead of the third one we will rather consider the subcategory

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})$$

of complexes of sheaves set-theoretically supported on the preimage of the nilpotent cone in $\mathfrak{g}^\vee$ (i.e. whose restriction to the complement vanishes).

The Grothendieck groups of all these categories identify with the group algebra of W_{ext}.

1.3. **Convolution of coherent sheaves.** We start with a baby example. Consider finite sets X, Y, and a map $f : X \to Y$. Then the vector space $\mathcal{F}(X \times_Y X)$ of $\mathbb{C}$-valued functions on the finite set $X \times_Y X = \{(a,b) \in X^2 \mid f(a) = f(b)\}$ has an associative product defined by the formula

$$(f \cdot g)(a,c) = \sum_{\substack{b \in X \\ f(b)=f(a)=f(c)}} f(a,b) \cdot f(b,c)$$

for $a, c \in X$ such that $f(a) = f(c)$.

Now we upgrade this to a categorical framework. Consider data $(M_z : z \in X \times_Y X)$ where each M_z is a complex of $\mathbb{C}$-vector spaces. There is a monoidal product on the category of such data defined as follows. Given collections $M = (M_z)_z$ and $M' = (M'_z)_z$, the product $M'' = M \star M'$ satisfies

$$M''_{(a,c)} = \bigoplus_{\substack{b \in X \\ f(b)=f(a)=f(c)}} M_{(a,b)} \otimes_{\mathbb{C}} M'_{(b,c)}.$$

Finally we consider the setting we will really use. Consider two schemes X, Y and a morphism $f : X \to Y$. We have three maps

$$p_{1,2}, p_{2,3}, p_{1,3} : X \times^R_Y X \times^R_Y X \to X \times^R_Y X$$

where $p_{i,j}$ is the projection on the i-th and j-th factors. The category $D\mathsf{QCoh}(X \times^R_Y X)$ has a convolution product defined by the formula

$$\mathscr{F} \star \mathscr{G} = R(p_{1,3})_* \left(L(p_{1,2})^* \mathscr{F} \otimes^L_{\mathscr{O}_{X \times^R_Y X \times^R_Y X}} L(p_{2,3})^* \mathscr{G} \right).$$

The monoidal unit for this product is $\Delta_* \mathscr{O}_X$ where $\Delta : X \to X \times^R_Y X$ is the diagonal embedding.

Such a formula makes sense also in an equivariant context, and we deduce:

- a monoidal product $\star$ on the category $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$;
- a monoidal product $\star$ on the category[2] $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})$;
- a left action of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})$ on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$;
- a right action of $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$ on $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$.

1.4. **Statement.** Bezrukavnikov constructed in [2] a series of three equivalences of triangulated categories:

$$\Phi_{I,I} : D^{\mathrm{b}}(I \backslash \mathrm{Fl}_G) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}),$$
$$\Phi_{I_u,I} : D^{\mathrm{b}}(I_u \backslash \mathrm{Fl}_G) \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}),$$
$$\Phi_{I_u,I_u} : D^{\mathrm{b}}_{I_u,I_u} \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}).$$

Here:

- $\Phi_{I,I}$ is an equivalence of monoidal categories;
- Φ_{I_u,I_u} is an equivalence of monoidal categories;
- $\Phi_{I_u,I}$ is an equivalence of bimodules categories, in the sense that for $\mathscr{F}$ in $D^{\mathrm{b}}_{I_u,I_u}$, $\mathscr{G}$ in $D^{\mathrm{b}}(I_u \backslash \mathrm{Fl}_G)$ and $\mathscr{H}$ in $D^{\mathrm{b}}(I \backslash \mathrm{Fl}_G)$ we have
$$\Phi_{I_u,I}(\mathscr{F} \star^{I_u} \mathscr{G} \star^I \mathscr{H}) \cong \Phi_{I_u,I_u}(\mathscr{F}) \star \Phi_{I_u,I}(\mathscr{G}) \star \Phi_{I,I}(\mathscr{H}).$$

2. Some ideas from the proof

2.1. **Compatibilities.**

2.1.1. *Between the three equivalences.* The fact that $\Phi_{I_u,I}$ is an equivalence of bimodule categories implies that:

- the forgetful functor $D^{\mathrm{b}}(I \backslash \mathrm{Fl}_G) \to D^{\mathrm{b}}(I_u \backslash \mathrm{Fl}_G)$ corresponds to the pushforward functor $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$;
- the pushforward functor $D^{\mathrm{b}}_{I_u,I_u} \to D^{\mathrm{b}}(I_u \backslash \mathrm{Fl}_G)$ corresponds to the (derived) pullback functor $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}) \to D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$.

2.1.2. *With the geometric Satake equivalence, the central functor and Wakimoto sheaves.* Consider the geometric Satake equivalence

$$\mathsf{Sat} : \mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{Rep}(G^\vee).$$

Then for $V \in \mathrm{Rep}(G^\vee)$ we have

$$\Phi_{I,I}(\mathsf{Z}(\mathsf{Sat}^{-1}(V))) = V \otimes \mathscr{O}_{\Delta \widetilde{\mathcal{N}}}.$$

For any $\lambda \in X_*(T) = X^*(T^\vee)$ we have a line bundle $\mathscr{O}_{G^\vee/B^\vee}(\lambda)$ on $G^\vee/B^\vee$. (It corresponds to the 1-dimensional $B^\vee$-module $\mathbb{C}_{B^\vee}(\lambda)$ under the equivalence

[2] In fact, in this case there is no monoidal unit.

$\mathsf{Coh}^{G^\vee}(G^\vee/B^\vee) = \mathrm{Rep}(B^\vee)$.) We denote by $\mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda)$ its pullback to $\widetilde{\mathcal{N}}$. Then we have

$$\Phi_{I,I}(\mathscr{W}_\lambda) = \Delta_* \mathscr{O}_{\widetilde{\mathcal{N}}}(\lambda).$$

The Wakimoto filtration of $\mathsf{Z}(\mathsf{Sat}^{-1}(V))$ corresponds on the dual side to the following fact: for $V \in \mathrm{Rep}(G^\vee)$, the coherent sheaf $V \otimes \mathscr{O}_{G^\vee/B^\vee}$ admits a canonical filtration with subquotients $\mathscr{O}_{G^\vee/B^\vee}(\lambda)$ where λ runs over the weights of V, and the line bundle $\mathscr{O}_{G^\vee/B^\vee}(\lambda)$ appears with multiplicity the dimension of the λ-weight space V_λ of V. (This comes from the filtration of V as a $B^\vee$-module with subquotients $\mathbb{C}_{B^\vee}(\lambda)$ where λ runs over the weights of V, with $\mathbb{C}_{B^\vee}(\lambda)$ appearing with multiplicity $\dim(V_\lambda)$.

2.1.3. *With the Arkhipov–Bezrukavnikov equivalence.* Recall the Iwahori–Whittaker derived category $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G)$, and the equivalence of categories

$$\Phi_{\mathcal{IW},I} : D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G) \cong D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$$

from [1], discussed in Lecture 16. Here $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G)$ is naturally a right module for the category $D^{\mathrm{b}}(I\backslash \mathrm{Fl}_G)$, and $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}})$ is naturally a right module for $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}(\widetilde{\mathcal{N}} \times^R_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}})$. For $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G)$ and $\mathscr{G}$ in $D^{\mathrm{b}}(I\backslash \mathrm{Fl}_G)$ we have

$$\Phi_{\mathcal{IW},I}(\mathscr{F} \star^I \mathscr{G}) \cong \Phi_{\mathcal{IW},I}(\mathscr{F}) \star \Phi_{I,I}(\mathscr{G}).$$

In [2], Bezrukavnikov also considers the subcategory $D_{\mathcal{IW},I_u}$ of the Iwahori–Whittaker derived category of sheaves on $\widetilde{\mathrm{Fl}}_G$ generated by objects obtained by pullback from $D^{\mathrm{b}}_{\mathcal{IW}}(\mathrm{Fl}_G)$, and constructs (using the same strategy as in [1]) an equivalence of triangulated categories

$$\Phi_{\mathcal{IW},I_u} : D^{\mathrm{b}}_{\mathcal{IW},I_u} \cong D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee})$$

where the right-hand side is the derived category of equivariant coherent sheaves on $\widetilde{\mathfrak{g}^\vee}$ supported on $\widetilde{\mathcal{N}}$. This equivalence is compatible with the actions of $D^{\mathrm{b}}_{I_u,I_u}$ and $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})$ via Φ_{I_u,I_u}.

2.2. **Rough idea of the construction of the equivalence.** The work lies in the construction of the equivalence Φ_{I_u,I_u}; the other versions are then obtained by rather formal procedures.

The categories $D^{\mathrm{b}}_{I_u,I_u}$ and $D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}_{\mathcal{N}}(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})$ "do not have enough objects" for the purpose of construction of the functor Φ_{I_u,I_u}, and one needs to consider some kind of "completions." (More specifically, the objects that one would like to use are the *tilting perverse sheaves*, and those will only exist in this completed picture.) On the constructible side one considers some category $D^\wedge_{I_u,I_u}$ of pro-objects in $D^{\mathrm{b}}_{I_u,I_u}$, whose theory was established by Yun. On the constructible side one works with a certain completion $(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})^\wedge$ of the scheme $\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}$. So one needs to construct an equivalence of triangulated categories

$$\Phi^\wedge_{I_u,I_u} : D^\wedge_{I_u,I_u} \xrightarrow{\sim} D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}((\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})^\wedge).$$

One also has a completed version $(\widetilde{\mathfrak{g}^\vee})^\wedge$ of the scheme $\widetilde{\mathfrak{g}^\vee}$, and a diagonal embedding

$$\delta : (\widetilde{\mathfrak{g}^\vee})^\wedge \to (\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})^\wedge.$$

As a first step in his construction, Bezrukavnikov constructs a functor

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}((\widetilde{\mathfrak{g}^\vee})^\wedge) \to D^\wedge_{I_u,I_u} \tag{2.1}$$

by essentially repeating some constructions from [1].

Remark 2.1. (1) A posteriori, this functor will correspond to $(\Phi^{\wedge}_{I_u,I_u})^{-1} \circ \delta_*$.

(2) There is also a completed version $D^{\wedge}_{\mathcal{IW},I_u}$ of the category $D^{\mathrm{b}}_{\mathcal{IW},I_u}$, and a "Whittaker averaging" functor

$$D^{\wedge}_{I_u,I_u} \to D^{\wedge}_{\mathcal{IW},I_u}. \tag{2.2}$$

The composition

$$D^{\mathrm{b}}\mathsf{Coh}^{G^\vee}((\widetilde{\mathfrak{g}^\vee})^\wedge) \xrightarrow{(2.1)} D^{\wedge}_{I_u,I_u} \xrightarrow{(2.2)} D^{\wedge}_{\mathcal{IW},I_u}$$

is an equivalence of categories, which is a "completed version" of the equivalence $(\Phi_{\mathcal{IW},I_u})^{-1}$.

Then this functor is "extended" to a functor

$$D^{\mathrm{b}}_{\mathrm{perf}}\mathsf{Coh}^{G^\vee}((\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})^\wedge) \to D^{\wedge}_{I_u,I_u} \tag{2.3}$$

(where the left-hand side is the subcategory of perfect complexes) using the requirement that the structure sheaf of $(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})^\wedge$ should correspond to the "completed big tilting object."

Remark 2.2. (1) The completed big tilting object is a categorical incarnation of the antisymmetrizer $\xi = \sum_{w\in W}(-1)^{\ell(w)}w$ in the group algebra of the Weyl group W. From this point of view, $D^{\mathrm{b}}_{\mathrm{perf}}\mathsf{Coh}^{G^\vee}((\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})^\wedge)$ corresponds to the 2-sided ideal in the affine Hecke algebra generated by ξ.

(2) Bezrukavnikov informally explains the roles of the various ingredients in the construction of (2.3) from the point of view of "category over a stack" in [2, §2.2]. (In this sense, a triangulated category *over a stack* S is a triangulated category equipped with an action of the monoidal category of perfect complexes on S.) Namely one consider the following stacks:

$$(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})/G^\vee \rightrightarrows \widetilde{\mathfrak{g}^\vee}/G^\vee \to \mathfrak{g}^\vee/G^\vee \to \mathrm{pt}/G^\vee.$$

The central functor Z allows to see $D^{\wedge}_{I_u,I_u}$ as a category over $\mathrm{pt}/G^\vee$. The monodromy operation on central sheaves allows to extend this to a category over $\mathfrak{g}^\vee/G^\vee$. Then the filtration of central sheaves by Wakimoto objects allows to lift this further to a category $\widetilde{\mathfrak{g}^\vee}/G^\vee$. Then one uses multiplication on both sides, and the fact that central objects are central, to finally lift the structure to a category over $(\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee})/G^\vee$. Once this is done, (2.3) corresponds to the action on the completed big tilting object.

Finally, this functor is used to construct the functor $\Phi^{\wedge}_{I_u,I_u}$ (in the reverse direction) using a general result characterizing, for X an appropriate algebraic stack, the category $D^{\mathrm{b}}\mathsf{Coh}(X)$ as a subcategory of the category of functors from $D^{\mathrm{b}}_{\mathrm{perf}}(X)$ to vector spaces.

References

[1] S. Arkhipov and R. Bezrukavnikov, *Perverse sheaves on affine flags and Langlands dual group* (with an appendix by R. Bezrukavnikov and I. Mirković), Israel J. Math. **170** (2009), 135–183.

[2] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*, Publ. Math. IHES **123** (2016), 1–67.