

Applications ①

I. Motivation for Q. gps

G - algebraic gp.

$$\mathfrak{g} = \text{Lie}(G) = T_1 G = (\mathfrak{m}/\mathfrak{m}^2)^*$$

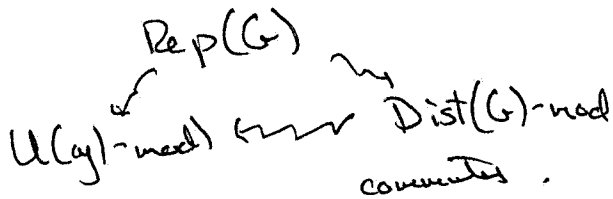
$\mathfrak{m} = \text{max. ideal at } 1.$

Two ways to make a ring

$U(\mathfrak{g})$
univ. env. alg.

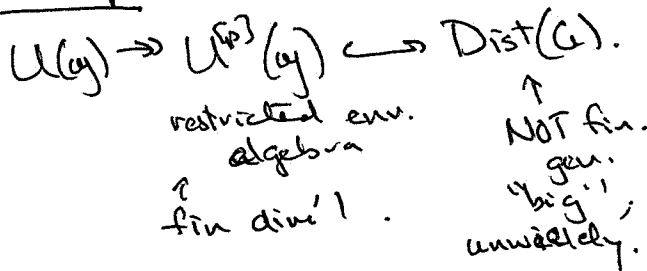
$\text{Dist}(G)$
built from $(\mathfrak{m}/\mathfrak{m}^2)^*$ $n \geq 0$.

Flat map $U(\mathfrak{g}) \rightarrow \text{Dist}(G)$.



Over \mathbb{C} : $U(\mathfrak{g}) \cong \text{Dist}(G)$.
↑ not useful.

Over $\overline{\mathbb{F}_p}$:

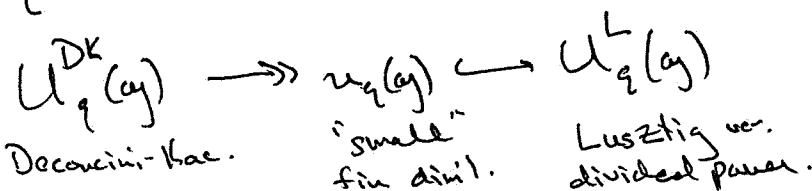


Quantum groups.

$U_q(\mathfrak{g})$: deform $U(\mathfrak{g})$ with a parameter $q \in \mathbb{C}$.

q "generic": rep theory same as for $U(\mathfrak{g})$ over \mathbb{C} .

q a root of unity: 3 versions:



Generators & relations: boring.

II. Cohomology

Thm (~1984, Andersen-Jantzen, Friedlander-Parshall)

$$G/k = \overline{\mathbb{F}_p}, p \text{ not too small.}$$

$$\text{Ext}_{U_{\overline{\mathbb{F}_p}}(\mathfrak{g})}^i(k, k) = k[\mathcal{N}]$$

nilp. case.

Thm (1993, Ginzburg-Kuran)

$$\text{Ext}_{u_{\mathbb{C}}(\mathfrak{g})}^i(\mathbb{C}, \mathbb{C}) = \mathbb{C}[\mathcal{N}].$$

More generally: \mathbb{C} any module
 $\text{Ext}_{u_{\mathbb{C}}(\mathfrak{g})}^i(\mathbb{C}, M)$ is finger

($\mathbb{C}[\mathcal{N}]$ -module = coh. sheaf on \mathcal{N} .)

Call this $H^*(u_q(\mathfrak{g}), M)$
 = cohomology of the small q -gp w/ coeffs in M .

Fact. If M is a $U_q^L(\mathfrak{g})$ -module
 $H^*(M)$ is a G -equiv. coh. sheaf.

What is it?

1990s Seemed too hard, but Humphreys proposed conj for

Supp. H^i (tilting $U_q^L(\mathfrak{g})$ -module)

↳ a G -orbit closure incl. based on KL combinatorics.

Applications (2)

III. Solution (Bezrukavnikov +)

$\text{Repo}(U_q^L(\mathfrak{g}))$ - "principal block" of $U_q^L(\mathfrak{g})$ -modules

- gen by modules in the linkage class of \mathbb{C} -triv. module.

"Translation functors"

- Known since ~~1970~~ 1970s for \mathfrak{g}
- move between blocks
- compose to get endofunctors of prin. block.

ring gen by these endofun. $\hookrightarrow K(\text{Repo}(U_q^L(\mathfrak{g})))$

Growth. gp
↓
antisph. modules

What are they?
↓
 $\mathbb{Z}[\text{Waff}]$

Suggests -

$\text{Repo}(U_q^L(\mathfrak{g}))$ related to $\text{Coh}^{G \times G_m}(\tilde{N})$,
 $D_{\text{IW}}^b(\text{Fl})$.

Thm. (Arkhipov - Bez. - Ginz.)

1) \exists "degrading functor"

$$D_{\text{Coh}}^b G \times G_m(\tilde{N}) \rightarrow D_{\text{Repo}}^b(U_q^L(\mathfrak{g})).$$

2) The comp.

$$D_{\text{IW}}^b(\text{Fl}) \simeq D_{\text{Coh}}^b G \times G_m(\tilde{N}) \rightarrow D_{\text{Repo}}^b$$

sends IC's \longleftarrow tilting modules.

Thm (Bezrukavnikov)

$$\begin{array}{ccc} \textcircled{1} & D_{\text{Coh}}^b G \times G_m(\tilde{N}) & \rightarrow D_{\text{Repo}} \\ & \downarrow & \downarrow H^*(a_q, -) \\ & D_{\text{Coh}}^b G \times G_m(\tilde{N}) & \xrightarrow{H^*} \text{Coh}(\mathcal{N}) \end{array}$$

commutes.

$$\begin{array}{ccc} \textcircled{2} & D_{\text{IW}}^b(\text{Fl}) & \rightarrow D_{\text{Coh}}^b G \times G_m(\tilde{N}) \\ & \searrow \text{sends IC's to } \mathbb{O} \text{ or perov-coherent IC "dC"} & \downarrow \\ & & D_{\text{Coh}}^b G \times G_m(\tilde{N}) \end{array}$$

Cor.

$H^*(U_q(\mathfrak{g}))$, tilting $U_q^L(\mathfrak{g})$ module

$$= H^*(\text{certain dC on } \mathcal{N})$$

can specify explicitly with KL combinatorics.

Combine w/ other thms on KL combinatorics \Rightarrow

Cor. Humphrey's conj. is true.