Applications ①

I. Motivation for Q. gps

G - algebraic gp.

$\mathfrak{g} = \mathrm{Lie}(G) = T_1 G = (m/m^2)^*$

m = max. ideal at 1.

Two ways to make a ring

$U(\mathfrak{g})$ — univ. env. alg.

$\mathrm{Dist}(G)$ — built from $(m/m^n)^*$ $n \geq 0$.

$\exists$ nat'l map $U(\mathfrak{g}) \to \mathrm{Dist}(G)$.

$$\begin{array}{ccc} & \mathrm{Rep}(G) & \\ \swarrow & & \searrow \\ U(\mathfrak{g})\text{-mod} & \leftsquigarrow & \mathrm{Dist}(G)\text{-mod} \end{array}$$

commutes.

<u>Over $\mathbb{C}$</u>: $U(\mathfrak{g}) \simeq \mathrm{Dist}(G)$. not useful.

<u>Over $\mathbb{F}_p$</u>:

$U(\mathfrak{g}) \twoheadrightarrow U^{[p]}(\mathfrak{g}) \hookrightarrow \mathrm{Dist}(G)$.

$U^{[p]}(\mathfrak{g})$: restricted env. algebra; fin dim'l.

$\mathrm{Dist}(G)$: NOT fin. gen. "big", unwieldy.

<u>Quantum groups</u>.

$U_q(\mathfrak{g})$: deform $U(\mathfrak{g})$ with a parameter q. $/\mathbb{C}$

q "generic": rep theory same as for $U(\mathfrak{g})$ over $\mathbb{C}$.

q a root of unity: 3 versions:

$U_q^{DK}(\mathfrak{g}) \twoheadrightarrow u_q(\mathfrak{g}) \hookrightarrow U_q^L(\mathfrak{g})$

De Concini-Kac. "small" fin dim'l. Lusztig ver. divided power.

Generators & relations: boring.

II. Cohomology

<u>Thm</u> (~1984, Andersen-Jantzen, Friedlander-Parshall)

$G / \Bbbk = \overline{\mathbb{F}_p}$, p not too small.

$\mathrm{Ext}^\bullet_{U^{[p]}(\mathfrak{g})}(\Bbbk, \Bbbk) = \Bbbk[\mathcal{N}]$ — nilp. cone.

<u>Thm</u> (1993, Ginzburg-Kumar)

$\mathrm{Ext}^\bullet_{u_q(\mathfrak{g})}(\mathbb{C}, \mathbb{C}) = \mathbb{C}[\mathcal{N}]$.

More generally: any module

$\mathrm{Ext}^\bullet_{u_q(\mathfrak{g})}(\mathbb{C}, M)$ is fin gen

a $\mathbb{C}[\mathcal{N}]$-module

= coh. sheaf on $\mathcal{N}$.

Call this $H^\bullet(u_q(\mathfrak{g}), M)$

= "cohomology of the small q. gp w/ coeffs in M".

<u>Fact</u>. If M is a $U_q^L(\mathfrak{g})$-module, $H^\bullet(M)$ is a G-eqvt coh. sheaf.

What is it?

<u>1990s</u> Seemed too hard, but Humphreys proposed conj for

Supp. $H^\bullet$(tilting $U_q^L(\mathfrak{g})$-module)

a G-orbit closure in $\mathcal{N}$, based on KL combinatorics

Applications (2)

III. Solution (Bezrukavnikov +).

$\mathrm{Rep}_0(U_q^L(\mathfrak{g}))$ - "principal block" of $U_q^L(\mathfrak{g})$-modules

- gen by modules in the linkage class of $\mathbb{C}$ = triv module.

"Translation functors"

- Known since ~~1970~~ 1970s for $\mathfrak{g}$
- move between blocks
- compose to get endofunctors of prin. block.

$$\underbrace{\text{ring gen by these endofun.}}_{\text{What are they? } \mathbb{Z}[W_{aff}]} \circlearrowright \underbrace{K(\mathrm{Rep}_0(U_q^L(\mathfrak{g})))}_{\substack{\text{Groth. gp} \\ \downarrow \\ \text{antisph. module}}}$$

Suggests:

$\mathrm{Rep}_0(U_q^L(\mathfrak{g}))$ related to $\mathrm{Coh}^{G\times\mathbb{C}^*}(\tilde{\mathcal{N}})$, $D^b_{IW}(\mathcal{F}\ell)$.

Thm. (Arkhipov - Bez. - Ginz).

1) $\exists$ "degrading functor"

$$D^b\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}) \longrightarrow D^b\mathrm{Rep}_0(U_q^L(\mathfrak{g})).$$

2) The comp.

$$D^{mix}_{IW}(\mathcal{F}\ell) \simeq D^b\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}) \to D^b\mathrm{Rep}_0$$

sends IC's $\longmapsto$ tilting modules.

Thm (Bezrukavnikov

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$$\begin{tikzcd}
D^b\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[r] \arrow[d] & D^b\mathrm{Rep}_0 \arrow[d, "H^\bullet(u_q{,}-)"] \\
D^b\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathcal{N}) \arrow[r, "H^\bullet"] & \mathrm{Coh}(\mathcal{N})
\end{tikzcd}$$

commutes.

②
$$\begin{tikzcd}
D^b_{IW}(\mathcal{F}\ell) \arrow[r] & D^b\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[d] \\
& D^b\mathrm{Coh}^{G\times\mathbb{G}_m}(\mathcal{N})
\end{tikzcd}$$

sends IC's to 0 or perv.-coherent IC "$\mathcal{IC}$"

Cor.

$H^\bullet(u_q(\mathfrak{g})$, tilting $U_q^L(\mathfrak{g})$-modules)

$= H^\bullet(\text{certain } \mathcal{IC} \text{ on } \mathcal{N})$

can specify explicitly with KL combinatorics.

Combine w/ other thms on KL combinatorics $\Rightarrow$

Cor. Humphrey's conj. is true.