# LECTURE 2: THE SATAKE ISOMORPHISM

## 1. Representations of $p$-adic groups

1.1. **Basic data.** We fix a (non archimedian) local field $F$, let $O_F$ be its ring of integers, and $k_F$ be its residue field. We also fix a uniformizer $\varpi \in O_F$ and denote by $q$ the cardinality of $k_F$. What these words precisely mean will not be important; the reader should rather keep in mind the following examples:

- $F = \mathbb{Q}_p$, $O_F = \mathbb{Z}_p$, $\varpi = p$; here $k_F = \mathbb{F}_p$ and $q = p$;
- $F = \mathbb{F}_q((t))$, $O_F = \mathbb{F}_q[[t]]$, $\varpi = t$; here $k_F = \mathbb{F}_q$.

We also fix a split reductive group scheme $\underline{G}$ over $O_F$ and set $\mathcal{G} = \underline{G}(F)$, $\mathcal{K} = \underline{G}(O_F)$. Then $\mathcal{G}$ has a canonical structure of topological group (it is locally compact and totally disconnected) and $\mathcal{K}$ is a compact open subgroup. Once again, we will not give formal definitions, but the reader can think of the examples $\underline{G} \in \{\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n\}$, with the topology of $\mathcal{G}$ and $\mathcal{K}$ induced by that of $F$ (i.e. a sequence of matrices converges iff all the entries in these matrices converge).

1.2. **Quick reminder on Haar measures.** In the representation of finite groups, one often uses the ability to sum over elements in the group: think of the proof of Maschke's theorem, or the (related) fact that the element $e_G = \frac{1}{|G|}\sum_{g \in G} g$ in the group algebra $\mathbb{C}[G]$ satisfies $e_G \cdot h = h \cdot e_G = e_G$ for any $h \in G$. When the group is not finite the sum doesn't make sense anymore. But when the group has a topology and is *locally compact*, there is a replacement given by the (left) Haar measure. Namely, there exists a unique (up to positive scalar) measure $\mathrm{d}_G$ on $G$ such that $\mathrm{d}_G(hB) = \mathrm{d}_G(B)$ for any Borel set $B \subset G$. This measure is finite on compact subsets of $G$, so that one can integrate compactly supported continuous functions, and we therefore have

$$\int_G f(hg)\mathrm{d}_G(g) = \int_G f(g)\mathrm{d}_G(g)$$

for any such function $f$ and any $h \in G$. When $G$ is compact, the measure can be normalized so that $\mathrm{d}_G(G) = 1$. (In case $G$ is finite, one recovers the measure so that each element has weight $\frac{1}{|G|}$.)

In particular, on $\mathcal{G}$ we have a left Haar measure $\mathrm{d}_{\mathcal{G}}$, which we normalize so that $\mathrm{d}_{\mathcal{G}}(\mathcal{K}) = 1$.

There is similarly a *right* Haar measure, which satisfies the similar condition as above but now with respect to multiplication *on the right* by an element on the group. A group is said to be *unimodular* if the left and right Haar measure coincide. This class includes all compact groups, and also the groups $\mathcal{G}$ as above. In particular, $\mathrm{d}_{\mathcal{G}}$ is also right invariant.

1.3. **Smooth representations.** We are interested in *smooth* representations of the group $\mathcal{G}$ on complex vector spaces, i.e. representations on a (not necessarily finite-dimensional) complex vector space $V$ such that the stabilizer of any vector in $V$ is open. These objects have been intensively studied but are complicated;

as a first step one can consider the *unramified* smooth representations, i.e. those which are generated (as representations of $\mathcal{G}$) by $V^{\mathcal{K}}$. (If $V$ is irreducible, this is equivalent to $V^{\mathcal{K}}$ being nonzero.) These representations can be described using a *Hecke algebra*, as follows.

Denote by

$$\mathsf{H}_{\mathcal{G}}$$

the space of compactly supported locally constant functions $f : \mathcal{G} \to \mathbb{C}$ which are $\mathcal{K}$-biinvariant, i.e. satisfy $f(kgk') = f(g)$ for any $g \in \mathcal{G}$ and $k, k' \in \mathcal{K}$. (This ring is sometimes called the "spherical affine Hecke algebra" attached to $\underline{G}$.)

If $V$ is a smooth representation of $\mathcal{G}$ then there is a natural operation

$$\mathsf{H}_{\mathcal{G}} \times V^{\mathcal{K}} \to V^{\mathcal{K}}$$

given by

$$(f, v) \mapsto \int_{\mathcal{G}} f(g)(g \cdot v) \mathrm{d}_{\mathcal{G}}(g).$$

In fact the space $\mathsf{H}_{\mathcal{G}}$ admits an algebra structure, with product defined by

$$(f \cdot g)(x) = \int_{\mathcal{G}} f(z) g(z^{-1}x) \mathrm{d}_{\mathcal{G}}(z),$$

so that the operation above defines an $\mathsf{H}_{\mathcal{G}}$-module structure on $V^{\mathcal{K}}$. (The unit element is the characteristic function $1_{\mathcal{K}}$.) Moreover, this construction provides an equivalence of abelian categories

$$\{\text{unramified smooth representations of } \mathcal{G}\} \cong \mathsf{H}_{\mathcal{G}}\text{–Mod}.$$

So, to go further in the study of unramified smooth representations one needs to understand the ring $\mathsf{H}_{\mathcal{G}}$.

*Remark* 1.1. For some details on the correspondence between Hecke algebra modules and representations, see [2].

1.4. **Rough structure of the Hecke algebra.** From now on we fix a split maximal torus $\underline{T} \subset \underline{G}$, and denote by

- $W$ the Weyl group of $(\underline{G}, \underline{T})$;
- $\mathbf{X}^\vee$ the lattice of cocharacters of $\underline{T}$.

For any $\lambda \in \mathbf{X}^\vee$, seen as a group schemes morphism $\mathbb{G}_{\mathrm{m},O_F} \to \underline{T}$, we have an induced group morphism

$$F^\times = \mathbb{G}_{\mathrm{m},O_F}(F) \to \underline{G}(F) = \mathcal{G}.$$

The image of $\varpi$ under this map will be denoted $\varpi^\lambda$. The *Cartan decomposition* says that we have

$$\mathcal{G} = \bigsqcup_{\lambda \in \mathbf{X}^\vee / W} \mathcal{K} \cdot \varpi^\lambda \cdot \mathcal{K}.$$

(Here the coset $\mathcal{K} \cdot \varpi^\lambda \cdot \mathcal{K}$ only depends on the $W$-orbit of $\lambda$.) Using this decomposition one sees that

$$\mathsf{H}_{\mathcal{G}} = \bigoplus_{\lambda \in \mathbf{X}^\vee / W} \mathbb{C} \cdot 1_{\mathcal{K} \cdot \varpi^\lambda \cdot \mathcal{K}}.$$

It is also a classical fact that the algebra $\mathsf{H}_{\mathcal{G}}$ is commutative. (This statement is sometimes called "Gelfand's lemma.")

*Example* 1.2. Assume that $\underline{G} = \underline{T} = \mathbb{G}_{\mathrm{m},O_F}$ and $F = \mathbb{F}_q((t))$. In this case we have $\mathcal{G} = (\mathbb{F}_q((t)))^\times$ and $\mathcal{K} = (\mathbb{F}_q[[t]])^\times$. (Here $\mathcal{K}$ consists of power series with nonzero constant term.) The Cartan decomposition in this case boilds to down to the observation that

$$(\mathbb{F}_q((t)))^\times = \bigsqcup_{n\in\mathbb{Z}} t^n \cdot (\mathbb{F}_q[[t]])^\times.$$

Since $(\mathbb{F}_q((t)))^\times$ is commutative, a function on $(\mathbb{F}_q((t)))^\times$ is $(\mathbb{F}_q[[t]])^\times$-biinvariant if and only if it factors through the quotient

$$(\mathbb{F}_q((t)))^\times/(\mathbb{F}_q[[t]])^\times \cong \mathbb{Z}.$$

If we denote by $c_n$ the characteristic function of $t^n \cdot (\mathbb{F}_q[[t]])^\times$, then for $n, m, l \in \mathbb{Z}$ we have

$$(c_n \cdot c_m)(t^l) = \int_{(\mathbb{F}_q((t)))^\times} 1_{t^n\cdot(\mathbb{F}_q[[t]])^\times}(z) 1_{t^m\cdot(\mathbb{F}_q[[t]])^\times}(z^{-1}t^l)\mathrm{d}x,$$

so that this quantity is the measure of $t^n(\mathbb{F}_q[[t]])^\times \cap t^{l-m}(\mathbb{F}_q[[t]])^\times$. Now we have

$$t^n(\mathbb{F}_q[[t]])^\times \cap t^{l-m}(\mathbb{F}_q[[t]])^\times = \begin{cases} t^n(\mathbb{F}_q[[t]])^\times & \text{if } n = l-m; \\ \varnothing & \text{otherwise.} \end{cases}$$

Hence

$$c_n \cdot c_m = c_{n+m}.$$

In other words, we have

$$\mathsf{H}_{\mathcal{G}} = \mathbb{C}[x, x^{-1}]$$

as algebras.

More generally, if $\underline{G} = \underline{T}$ we have

$$\mathsf{H}_{\mathcal{G}} = \mathbb{C}[\mathbf{X}^\vee].$$

## 2. The Satake isomorphism

2.1. **Statement.** The *Satake isomorphism* describes $\mathsf{H}_{\mathcal{G}}$ for a general $\underline{G}$ as above: it provides an algebra isomorphism

$$\mathsf{H}_{\mathcal{G}} \xrightarrow{\sim} \mathbb{C}[\mathbf{X}^\vee]^W.$$

In particular, it follows that the simple $\mathsf{H}_{\mathcal{G}}$-modules are 1-dimensional, and classified by the quotient $T^\vee/W$ where $T^\vee$ is the complex torus with character lattice $\mathbf{X}^\vee$.

A survey of the proof of this result is given in [1]. For a more detailed description of this proof in the case of $\mathrm{GL}_n(\mathbb{Q}_p)$, see also [3]. For other useful references of this subject, see [2, 4].

2.2. **Construction of the morphism via constant term.** We now fix a Borel subgroup $\underline{B} \subset \underline{G}$ containing $\underline{T}$, and denote by $\underline{U}$ its unipotent radical, so that we have

$$\underline{B} = \underline{T} \ltimes \underline{U}.$$

This choice determines a system of positive roots, and we denote by $\rho \in \frac{1}{2}X^*(\underline{T})$ the half-sum of the positive roots.

Note that $\mathcal{T} = \underline{T}(F)$ fits in the general setting studied so far, with $\mathcal{K} \cap \mathcal{T} = \underline{T}(O_F)$, and that by Example 1.2 we have

$$\mathsf{H}_{\mathcal{T}} = \mathbb{C}[\mathbf{X}^\vee].$$

We consider the unique Haar measure $\mathrm{d}_{\mathcal{U}}$ on $\mathcal{U} := \underline{U}(F)$ such that $\underline{U}(O_F)$ has measure 1. (Here again, $\mathcal{U}$ is known to be unimodular, so that $\mathrm{d}_{\mathcal{U}}$ is invariant on both sides.) We denote by

$$\delta^{1/2} : \mathcal{T} \to \mathbb{C}$$

the unique $(\mathcal{K} \cap \mathcal{T})$-invariant function such that

$$\delta^{1/2}(\varpi^{\lambda}) = q^{-\langle \lambda, \rho \rangle}.$$

In fact, $\delta^{1/2}$ is a group morphism. For $n \in \mathbb{Z}$ we set $\delta^{n/2} = (\delta^{1/2})^n$.

The *Satake transform* is the map

$$\mathcal{S} : \mathsf{H}_{\mathcal{G}} \to \mathsf{H}_{\mathcal{T}}$$

given by

$$\mathcal{S}(f)(t) = \delta^{1/2}(t) \cdot \int_{\mathcal{U}} f(tu) \mathrm{d}_{\mathcal{U}}(u)$$

for $t \in \mathcal{T}$.

*Remark* 2.1. I don't know a convincing heuristic explanation for the appearance of the factor $\delta^{1/2}$ in this formula. In fact, one can replace this factor by any power of $\delta^{1/2}$ (including the 0-th power) and still get an algebra morphism. However, for other powers this morphism will not factor through the $W$-invariants (see §2.3 below for details). There are a number of formula in representation theory that involve $\rho$ (e.g. Weyl's character formula for compact Lie groups of reductive algebraic groups, Serre duality for line bundles on flag varieties, or the "dot-action" in the modular representation theory of reductive algebraic groups), and this is another example of this phenomenon.

The meaning of this formula can be explained as follows. Consider the space $\mathsf{M}$ of locally constant functions $f : \mathcal{G} \to \mathbb{C}$ which are left invariant under $\mathcal{K}$ and right invariant under $(\mathcal{K} \cap \mathcal{T}) \ltimes \mathcal{U}$, i.e. which satisfy

$$f(kxb) = f(x) \quad \text{for } x \in \mathcal{G},\ k \in \mathcal{K} \text{ and } b \in (\mathcal{K} \cap \mathcal{T}) \ltimes \mathcal{U},$$

and which moreover are nonzero only on finitely many double cosets for $\mathcal{K}$ (on the left) and $(\mathcal{K} \cap \mathcal{T}) \ltimes \mathcal{U}$ (on the right). This space is a left module for $\mathsf{H}_{\mathcal{G}}$ for the action defined by

$$(f \cdot g)(x) = \int_{\mathcal{G}} f(y^{-1}) g(yx) \mathrm{d}_{\mathcal{G}}(y)$$

for $f \in \mathsf{H}_{\mathcal{G}}$, $g \in \mathsf{M}$ and $x \in \mathcal{G}$, and a right module for $\mathsf{H}_{\mathcal{T}}$ for the action defined by

$$(g \cdot h)(x) = \int_{\mathcal{T}} g(xy) h(y^{-1}) \mathrm{d}_{\mathcal{T}}(y)$$

for $g \in \mathsf{M}$, $h \in \mathsf{H}_{\mathcal{T}}$ and $x \in \mathcal{G}$.

**Lemma 2.2.** *For any $f \in \mathsf{H}_{\mathcal{G}}$ and $g \in \mathsf{M}$ we have*

$$f \cdot g = g \cdot (\delta^{-1/2} \mathcal{S}(f)).$$

*Proof.* The proof will use the following identity, which follows from [1, Equations (5), (8) and (10)]:

$$\int_{\mathcal{G}} f(g) \mathrm{d}_{\mathcal{G}}(g) = \int_{\mathcal{K}} \int_{\mathcal{T}} \int_{\mathcal{U}} f(kut) \mathrm{d}_{\mathcal{G}}(k) \mathrm{d}_{\mathcal{T}}(t) \mathrm{d}_{\mathcal{U}}(u). \tag{2.1}$$

It is well known that we have a decomposition

$$\mathcal{G} = \bigsqcup_{\lambda \in \mathbf{X}^\vee} \mathcal{K} \cdot \varpi^\lambda \cdot \mathcal{U} \cdot (\mathcal{K} \cap \mathcal{T}) \tag{2.2}$$

(called the *Iwasawa decomposition*). Hence an element in $\mathsf{M}$ is characterized by its restriction to $\mathcal{T}$. Now if $x \in \mathcal{T}$, using (2.1) we see that

$$(f \cdot g)(x) = \int_{\mathcal{G}} f(y^{-1}) g(yx) \mathrm{d}_{\mathcal{G}}(y)$$
$$= \int_{\mathcal{K}} \int_{\mathcal{T}} \int_{\mathcal{U}} f(t^{-1}u^{-1}k^{-1}) g(kutx) \mathrm{d}_{\mathcal{G}}(k) \mathrm{d}_{\mathcal{T}}(t) \mathrm{d}_{\mathcal{U}}(u).$$

Since $f$ is right $\mathcal{K}$-invariant, $g$ is left $\mathcal{K}$-invariant and right $\mathcal{U}$-invariant, and $\mathcal{K}$ has volume 1, we deduce that

$$(f \cdot g)(x) = \int_{\mathcal{T}} \int_{\mathcal{U}} f(t^{-1}u^{-1}) g(tx) \mathrm{d}_{\mathcal{T}}(t) \mathrm{d}_{\mathcal{U}}(u).$$

On the other hand we have

$$g \cdot (\delta^{-1/2}\mathcal{S}(f)) = \int_{\mathcal{T}} g(xy) \delta^{-1/2}(y^{-1}) \mathcal{S}(f)(y^{-1}) \mathrm{d}_{\mathcal{T}}(y)$$
$$= \int_{\mathcal{T}} \int_{\mathcal{U}} g(xy) f(y^{-1}u) \mathrm{d}_{\mathcal{T}}(y) \mathrm{d}_{\mathcal{U}}(u) = \int_{\mathcal{T}} \int_{\mathcal{U}} g(yx) f(y^{-1}u^{-1}) \mathrm{d}_{\mathcal{T}}(y) \mathrm{d}_{\mathcal{U}}(u).$$

Comparing these formulas we conclude. □

The decomposition (2.2) shows that

$$\mathsf{M} = \bigoplus_{\lambda \in \mathbf{X}^\vee} 1_{\mathcal{K} \cdot \varpi^\lambda \cdot \mathcal{U} \cdot (\mathcal{K} \cap \mathcal{T})}.$$

It is easily seen that for $\lambda, \mu \in \mathbf{X}^\vee$ we have

$$1_{\mathcal{K} \cdot \varpi^\lambda \cdot \mathcal{U} \cdot (\mathcal{K} \cap \mathcal{T})} \cdot 1_{\varpi^\mu (\mathcal{K} \cap \mathcal{T})} = 1_{\mathcal{K} \cdot \varpi^{\lambda+\mu} \cdot \mathcal{U} \cdot (\mathcal{K} \cap \mathcal{T})}.$$

Hence $\mathsf{M}$ is free, hence in particular faithful, as a right $\mathsf{H}_{\mathcal{T}}$-module. This property and Lemma 2.2 imply that the assignment

$$f \mapsto \delta^{-1/2}\mathcal{S}(f)$$

is an algebra morphism, from which one deduces easily that $\mathcal{S}$ is an algebra morphism.

2.3. **Proof.** A more explicit version of the Satake isomorphism is as follows.

**Theorem 2.3.** *The map*

$$\mathcal{S} : \mathsf{H}_{\mathcal{G}} \to \mathsf{H}_{\mathcal{T}}$$

*induces an algebra isomorphism*

$$\mathsf{H}_{\mathcal{G}} \xrightarrow{\sim} (\mathsf{H}_{\mathcal{T}})^W$$

*where $W$ acts on $\mathsf{H}_{\mathcal{T}} = \mathbb{C}[\mathbf{X}^\vee]$ via its natural action on $\mathbf{X}^\vee$.*

To prove this statement, one first has to check that $\mathcal{S}$ takes values in $W$-invariants in $\mathsf{H}_{\mathcal{T}}$. Then, one considers the subset

$$\mathbf{X}^\vee_+ \subset \mathbf{X}^\vee$$

of dominant cocharacters. One has a basis

$$(1_{\mathcal{K}\cdot\varpi^\lambda\cdot\mathcal{K}} : \lambda \in \mathbf{X}^\vee_+)$$

of $\mathsf{H}_\mathcal{G}$, and a basis of $(\mathsf{H}_\mathcal{T})^W$ given by

$$(1_{\bigsqcup_{\mu\in W(\lambda)} \varpi^\mu\cdot(\mathcal{T}\cap\mathcal{K})} : \lambda \in \mathbf{X}^\vee_+)$$

of $(\mathsf{H}_\mathcal{T})^W$. To finish the proof it suffices to show that the matrix expressing the image of the first basis in terms of the second one is invertible. For this one considers the order on $\mathbf{X}^\vee$ such that $\lambda \preceq \mu$ iff $\mu - \lambda$ is a sum of positive coroots. Then it suffices to show that the matrix is lower triangular (with invertible entries on the diagonal) with respect to the restriction of this order to $\mathbf{X}^\vee_+$. For this one has to show the following:

(1) for any $\lambda \in \mathbf{X}^\vee_+$, $\mathcal{S}(1_{\mathcal{K}\cdot\varpi^\lambda\cdot\mathcal{K}})(\varpi^\lambda) \neq 0$;
(2) for any $\lambda, \mu \in \mathbf{X}^\vee_+$, if $\mathcal{S}(1_{\mathcal{K}\cdot\varpi^\lambda\cdot\mathcal{K}})(\varpi^\mu) \neq 0$ then $\mu \preceq \lambda$.

In fact, from the definition we see that for $\lambda \in \mathbf{X}^\vee_+$ and $\mu \in \mathbf{X}^\vee$ we have

$$\mathcal{S}(1_{\mathcal{K}\cdot\varpi^\lambda\cdot\mathcal{K}})(\varpi^\mu) = q^{-\langle\mu,\rho\rangle}\cdot\int_\mathcal{U} 1_{\mathcal{K}\cdot\varpi^\lambda\cdot\mathcal{K}}(\varpi^\mu u)\mathrm{d}_\mathcal{U}(u) = q^{-\langle\mu,\rho\rangle}\cdot\mathrm{d}_\mathcal{U}(\mathcal{U}\cap(\varpi^{-\mu}\cdot\mathcal{K}\varpi^\lambda\mathcal{K})).$$

Now it is known that

$$(\varpi^\mu\cdot\mathcal{U})\cap(\mathcal{K}\varpi^\lambda\mathcal{K}) \neq 0 \quad \Rightarrow \quad \mu \preceq \lambda,$$

which implies (2). And when $\lambda = \mu$ we have

$$\mathcal{U}\cap(\varpi^{-\lambda}\cdot\mathcal{K}\varpi^\lambda\mathcal{K}) = \varpi^{-\lambda}\cdot(\mathcal{K}\cap\mathcal{U})\cdot\varpi^\lambda.$$

We have

$$\mathrm{d}_\mathcal{U}(\varpi^{-\lambda}\cdot(\mathcal{K}\cap\mathcal{U})\cdot\varpi^\lambda) = \delta(\varpi^{-\lambda}) = q^{2\langle\lambda,\rho\rangle}$$

by [1, Formula (8)], hence

$$\mathcal{S}(1_{\mathcal{K}\cdot\varpi^\lambda\cdot\mathcal{K}})(\varpi^\lambda) = q^{\langle\lambda,\rho\rangle},$$

which shows (1).

For the details we refer to [1, §4.2].

## 3. Towards categorification

Consider the complex connected reductive group $G^\vee$ which is Langlands dual to $\underline{G}$. This means that $G^\vee$ has a maximal torus $T^\vee \subset G^\vee$ whose group of *characters* is $\mathbf{X}^\vee$. The Weyl group $W$ identifies with the Weyl group of $G^\vee$. Recall also that the Grothendieck group $K^0(\mathrm{Rep}(G^\vee))$ identifies with $(\mathbb{Z}[\mathbf{X}^\vee])^W$. By Chevalley's theorem, the simple modules for $G^\vee$ are classified by $\mathbf{X}^\vee_+$; for $\lambda \in \mathbf{X}^\vee$ we denote by $\mathrm{ch}_\lambda$ the character of the simple module associated with $\lambda$.

The elements $(\mathrm{ch}_\lambda : \lambda \in \mathbf{X}^\vee_+)$ form a $\mathbb{Z}$-basis of $(\mathbb{Z}[\mathbf{X}^\vee])^W$, hence a $\mathbb{C}$-basis of

$$\mathbb{C}\otimes_\mathbb{Z}(\mathbb{Z}[\mathbf{X}^\vee])^W \cong (\mathbb{C}[\mathbf{X}^\vee])^W = (\mathsf{H}_\mathcal{T})^W.$$

One might wonder what is the corresponding basis of $\mathsf{H}_\mathcal{G}$ under the isomorphism of Theorem 2.3. In fact, if we denote this basis by $(M_\lambda : \lambda \in \mathbf{X}^\vee_+)$, then from the proof sketched above we know that there exist coefficients $(d_{\mu,\lambda} : \lambda, \mu \in \mathbf{X}^+,\ \mu \preceq \lambda)$ such that

$$M_\lambda = \sum_{\substack{\mu\in\mathbf{X}^+,\\ \mu\preceq\lambda}} d_{\mu,\lambda}\cdot 1_{\mathcal{K}\varpi^\mu\mathcal{K}}.$$

Then the question above can be rephrased as asking for an interpretation of the coefficients $d_{\mu,\lambda}$.

This question was tackled by Lusztig in [5], where he showed that $d_{\mu,\lambda}$ can be expressed in terms of the value at $q$ of a certain *Kazhdan–Lusztig polynomial* attached to some elements (depending on $\lambda$ and $\mu$) in the extended affine Weyl group $W_{\mathrm{ext}}$; see e.g. [4, Proposition 4.4].[1] Since these polynomials were also known to compute dimensions of the stalks of certain simple perverse sheaves on the affine Grassmannian of $G$ (which is a geometric version of the quotient $\mathcal{G}/\mathcal{K}$), this was a first indication that representations of $G^\vee$ might be connected to perverse sheaves on the affine Grassmannian of $G$, and the starting point for the *geometric Satake equivalence*.

## References

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[1] Our $d_{\mu,\lambda}$ corresponds to $q^{-\langle\lambda,\rho\rangle} \cdot d_\lambda(\mu)$ in the notation of [4].