Affine Grassmannians ①

I. Affine Grass. for ~~SL_n~~, GL_n

$\mathcal{O} = \mathbb{C}[[t]]$ = ring of formal power series
$\mathcal{K} = \mathbb{C}((t))$ = field of formal Laurent series

Defn. A lattice in $\mathcal{K}^n$ is an $\mathcal{O}$-submodule (nec. free) of rank n.

Gr_{GL_n} = set of lattices in $\mathcal{K}^n$.

Goal: Give it a topology.

① Valuation Any lattice L has an $\mathcal{O}$-basis $v_1, \dots, v_n$.

$\det \begin{bmatrix} \uparrow & & \uparrow \\ v_1 & \dots & v_n \\ \downarrow & & \downarrow \end{bmatrix} \in \mathcal{K}^\times$, not well-defined, but lowest power of t in it is.

$v(L) = \min\{n \mid t^n \text{ occurs in } \det(\text{basis}) \text{ w/nonzero coeff}\}$.

$Gr^k :=$ lattices of valuation k

$Gr = \bigsqcup_{n \in \mathbb{Z}} Gr^n$

② Comparison with $L_0 := \mathcal{O}^n \subset \mathcal{K}^n$

Example $n=2$, $L = \text{span} \begin{bmatrix} t^{-5}+e^t \\ t\sin t \end{bmatrix}, \begin{bmatrix} 1 \\ t^3 \end{bmatrix}$

$v(L) = -2$.

$t^3 L_0 \subset L \subset t^{-5} L_0$

Easy: $\forall$ lattice L, $\exists N \geq 0$ $\underbrace{t^N L_0 \subset L \subset t^{-N} L_0}$.

$Gr^{k,N} :=$ lattices of val. k with

Note: $\dfrac{t^{-N}L_0}{t^N L_0} \simeq \mathbb{C}^{2nN}$, $\circlearrowleft$ mult. by t induces a nilp. endo, $\bar{t}$.

$\dim \dfrac{L}{t^N L_0} = nN - v$.

$Gr^{k,N} = \{\bar{t}\text{-stable subspaces of dim } nN - v \text{ in } \mathbb{C}^{2nN}\}$

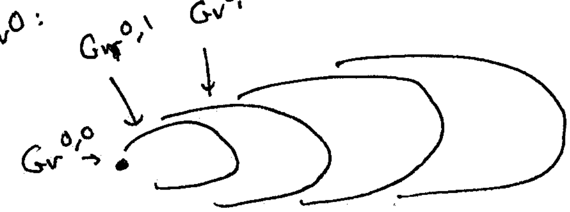
$\xhookrightarrow[\text{closed}]{} Gr(2nN, nN - v)$
↑ ordinary Grassmannian.

③ Topology.

1. Give $Gr^{k,N}$ subspace topology.
2. Note: $Gr^{k,N} \hookrightarrow Gr^{k,N+1}$. Show: closed embedding.
3. Give $Gr^k = \bigcup_N Gr^{k,N}$ the colimit topology.
4. ~~Gr~~ Give $Gr = \bigsqcup Gr^k$ disjoint union topology.

Example. $n=2$.

Gr^0:



dim: 0, 2, 4, . . .

$Gr^{0,1} \simeq \dfrac{\text{a } \mathbb{P}^1\text{-bdle over } \mathbb{P}^1}{\text{collapse one } \mathbb{P}^1 \text{ to a point}}$

④ Orbits

1. $GL_n(\mathcal{K}) \circlearrowright Gr$ transitively, stabilizer of L_0 is $GL_n(\mathcal{O})$
$\Rightarrow Gr \simeq GL_n(\mathcal{K}) / GL_n(\mathcal{O})$
2. $GL_n(\mathcal{O})$-orbits on Gr
$\xleftrightarrow{\sim} X^+ = \{(\lambda_1 \geq \dots \geq \lambda_n)\} \subseteq \mathbb{Z}^n$

orbit representatives:

$L_\lambda = \text{span} \begin{bmatrix} t^{\lambda_1} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ t^{\lambda_2} \\ \\ \end{bmatrix}, \dots, \begin{bmatrix} \\ \\ t^{\lambda_n} \end{bmatrix}$

Examples, fin dim'l, etc.

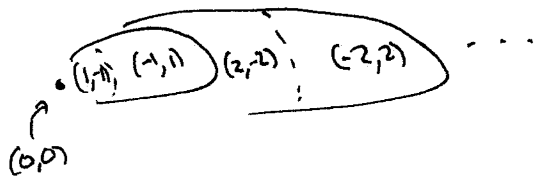
3. $I = \begin{bmatrix} \diagdown & t\mathcal{O} \\ \mathcal{O} & \diagdown \end{bmatrix} \subset GL_n(\mathcal{O})$.

I-orbits $\xleftrightarrow{\sim} X = \mathbb{Z}^n$.

Examples, affine spaces.

Affine Grassmannians ②

I-orbits on Gr^0:



4. $N^-(\mathcal{K})$-orbits $\longleftrightarrow X$.

$\dim N^-(\mathcal{K})\cdot L_\nu \cap Gr_\lambda = \langle \rho, \lambda - \nu \rangle$.

II. Affine Grassmannian for general G.

G - reductive. $/\mathbb{C}$

~~Choose~~ Defn.

$Gr_G = G(\mathcal{K})/G(\mathcal{O})$.

① Topology.

Choose embedding $G \hookrightarrow GL_n$.

Then $G(\mathcal{O}) = G(\mathcal{K}) \cap GL_n(\mathcal{O})$.
inside $GL_n(\mathcal{K})$.

Identify
$Gr_G = G(\mathcal{K})$-orbit of L_0

- closed in Gr_{GL_n}.
- Give it induced topology.

Fact. Indep. of choice of $G \hookrightarrow GL_n$.

SL_n: Gr_{SL_n} = lattices of vol. 0.

Classical gps: lattices w/extra condition.

$I = ev^{-1}(B^-)$, $ev: G(\mathcal{O}) \to G$.

② Orbits

$G(\mathcal{O})$-orbits $\longleftrightarrow X^+$ = dom cowts.

$\dim = \langle 2\rho, \lambda \rangle$

I-orbits $\longleftrightarrow X$

$\dim = |\langle 2\rho, \lambda \rangle| - \delta_\lambda$,

δ_λ = length of shortest $w \in W$ st. $w\lambda \in -X^+$.

$N^-(\mathcal{K})$-orbits $\longleftrightarrow X$.