(Pre) Sheaves on topological spaces

1. (Pre) sheaves:

Reminder: sheaf is a presheaf s.t. sections satisfy basic local-global compatibilities.

<u>example</u> $X = \{x, y\}$ w/ discrete topology:

open sets

$$\begin{tikzcd}
 & X & \\
\{x\} \arrow[ur] & & \{y\} \arrow[ul] \\
 & \emptyset \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

presheaf:

$$\begin{tikzcd}
 & V_X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
V_x \arrow[dr, "\gamma"'] & & V_y \arrow[dl, "\delta"] \\
 & V_\emptyset &
\end{tikzcd}$$

$\boxed{\gamma\alpha = \delta\beta}$

sheaf:

$$\begin{tikzcd}
 & V_x \times V_y \arrow[dl, "pr"'] \arrow[dr, "pr"] & \\
V_x \arrow[dr] & & V_y \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

2. SES. of (pre)sheaves:

Reminder: for presheaves, sections on an open set is an exact functor:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

$$\Rightarrow \quad 0 \to \Gamma(U, \mathcal{A}) \to \Gamma(U, \mathcal{B}) \to \Gamma(U, \mathcal{C}) \to 0$$

for sheaves, kernels formed in presheaves are still sheaves (limits play well w/ limits), but cokernels need to be sheafified.

<u>example</u> On a top space X, constant presheaf $\underline{\mathbb{C}}^p_X$ given by

$$\Gamma(U, \underline{\mathbb{C}}^p_X) = \underset{\substack{\text{constant} \\ \text{fn's } U \to \mathbb{C}}}{\mathbb{C}}, \quad \text{restriction maps} = \mathrm{id}_{\mathbb{C}}.$$

$$\mathrm{Hom}_{\mathrm{Presh}}(\underline{\mathbb{C}}^p_X, -) \simeq \Gamma(X, -).$$

In particular $\underline{\mathbb{C}}^p_X$ is projective in Presheaves.

example Constant sheaf $\underline{\mathbb{C}}_X$ is sheafification of $\underline{\mathbb{C}}'$, given by

$$\Gamma(U, \underline{\mathbb{C}}_X) = \left\{ \begin{array}{c} \text{locally constant} \\ \text{fns} \quad f: U \to \mathbb{C} \end{array} \right\}$$ w/ natural restriction maps

$$\mathrm{Hom}_{sh}(\underline{\mathbb{C}}_X, -) \simeq \Gamma(X, -),$$

but no longer exact - encodes interesting local ⟷ global problems. still have:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

only:

$$0 \to \Gamma(U, \mathcal{A}) \to \Gamma(U, \mathcal{B}) \to \Gamma(U, \mathcal{C})$$

but get exactness in stalks, i.e. "local limit"

$$0 \to \varinjlim_{U \ni x} \Gamma(U, \mathcal{A}) \to \varinjlim_{U \ni x} \Gamma(U, \mathcal{B}) \to \varinjlim_{U \ni x} \Gamma(U, \mathcal{C}) \to 0.$$

$$\text{!!}$$

$$0 \longrightarrow \mathcal{A}_x \longrightarrow \mathcal{B}_x \longrightarrow \mathcal{C}_x \longrightarrow 0 \qquad \forall x \in X$$

example Consider $X = \mathbb{C}^\times$,

$$\mathcal{A} = \underline{\mathbb{C}}_X, \qquad \mathcal{B} = C^0_X, \qquad \mathcal{A} \hookrightarrow \mathcal{B}$$

↑ sheaf of locally constant $\mathbb{C}$-valued fn's

↑ sheaf of cts $\mathbb{C}$-valued fn's

tautological embedding

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}^p \to 0$$

cokernel in presheaves

vs.

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

cokernel in sheaves

On any $U \subset \mathbb{C}^\times$ simply connected, have well defined element $\log z$ in

$$\Gamma(U, \mathcal{C}^p) \quad \text{and hence} \quad \Gamma(U, \mathcal{C})$$

($\log z$ only defined up to adding $2\pi i \mathbb{Z}$)

These do not arise as restrictions of a global section of $\mathcal{C}^p$, but do as a global section of $\mathcal{C}$!

Moreover, on global sections we have

$$0 \to \Gamma(\mathbb{C}^\times, \underline{\mathbb{C}}) \to \Gamma(\mathbb{C}^\times, C^0) \to \Gamma(\mathbb{C}^\times, \mathcal{C})$$

and $\log z \in \Gamma(\mathbb{C}^\times, \mathcal{C})$ is <u>not</u> in image of $\Gamma(\mathbb{C}^\times, C^0)$.

remark In fact have:

$$0 \to \Gamma(\mathbb{C}^\times, \underline{\mathbb{C}}) \to \Gamma(\mathbb{C}^\times, C^0) \to \Gamma(\mathbb{C}^\times, e) \to \mathbb{C}\cdot \log z \to 0$$

and $\mathbb{C}\log z \xleftrightarrow{\sim} H^1(\mathbb{C}^\times, \underline{\mathbb{C}})$ 

(detecting nontrivial fundamental group!).

$$\left((\pi_1)_{ab} \otimes_{\mathbb{Z}} \mathbb{C}\right)^* \simeq \mathbb{C}\cdot \log z .$$

3. Closed and open games

Recall given a map $f: X \to Y$ of topological spaces, have

$$f^*: Sh(Y) \rightleftarrows Sh(X) : f_*$$

Properties: $\Gamma(U, f_*\mathcal{F}) = \Gamma(f^{-1}U, \mathcal{F})$

f_* left exact ($X \to pt$ recovers global sections)

f^* has a simple formula on stalks:

$$(f^* \mathcal{F})_x \simeq \mathcal{F}_{f(x)}$$

and relatedly is exact.

<u>example</u> $\quad i : \{x\} \hookrightarrow X \quad$ inclusion of a point

Then $\quad Sh\{x\} \simeq Vect.$

- i_* produces skyscraper sheaf:

$$(i_* V)_y = \begin{cases} V & x = y \\ 0 & \text{o.w.} \end{cases}$$

- i^* sends a sheaf $\mathcal{F}$ to its stalk $\mathcal{F}_x$ at x.

In particular $\quad i^* i_* \xrightarrow{\sim} id$; same for any closed embedding $\quad i : Z \to X.$

What about other direction?

$$0 \to j_! j^* \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow i_* i^* \mathcal{G} \to 0$$

$\uparrow$

extension by zero

<u>example</u> $X = \mathbb{R}$

$$Z = \{0\} \overset{i}{\hookrightarrow} \mathbb{R} \overset{j}{\hookleftarrow} \mathbb{R}^\times = U$$

- $Sh(Z) \simeq \mathrm{Vect}$
- given $V \in Sh(Z)$, consider its $*$-pushforward. $*$-stalks are as follows:

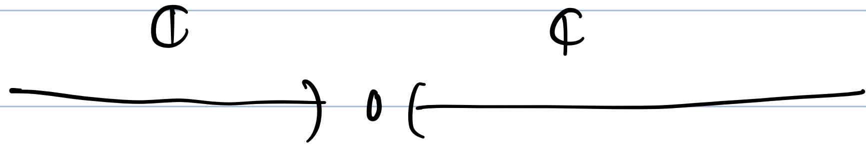
$$(i_* V)_t = \begin{cases} V & t = 0 \\ 0 & t \neq 0. \end{cases}$$

By contrast, given $\mathcal{G} \in Sh(U)$, have

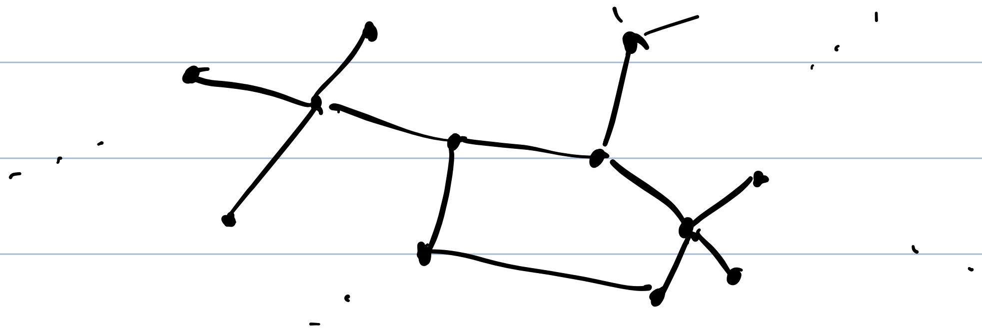
$$(j_* \mathcal{G})_t = \begin{cases} \mathcal{G}_t & t \neq 0, \\ \varinjlim_{U \ni 0} \Gamma(U \setminus 0, \mathcal{G}) & t = 0 \end{cases}$$

In particular

$$(j_* \underline{\mathbb{C}})_t = \begin{cases} \mathbb{C} & t \neq 0 \\ \mathbb{C}^2 & t = 0 \end{cases}$$



<u>exercise</u> Consider a (say planar) graph X



w/ analytic topology, $j : U = X \setminus \text{vertices} \hookrightarrow X$

compute stalks of $j_* \underline{\mathbb{C}}_U$.

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relatedly, $id \to j_* j^*$ not surjective, but still can form

$$0 \to i_* i^! \to id \to j_* j^*$$

sections supported on Z,

right adjoint to i_*.

exercise: Suppose $X = \mathbb{C}$,

$Z = \{0\}$, $i : Z \to X$

$U = \mathbb{C} \setminus \{0\}$; $j : U \to X$.

Compute all the stalks of $j_* \underline{\mathbb{C}}_U$.

bonus: Suppose $\mathcal{L}$ is a local system on U,
i.e. a sheaf which is locally isomorphic
to a constant sheaf. I.e., $\exists$ a cover V_i, $i \in I$ of
U, such that $\exists$ isomorphisms $\mathcal{L}|_{V_i} \simeq \underline{\mathbb{C}}^{\oplus r}_{V_i}$, for some $r \in \mathbb{Z}$.

a. show there is an equivalence of categories, for any point $u \in U$,

$$\left\{ \begin{array}{c} \text{local systems} \\ \text{on } U \end{array} \right\} \longrightarrow \left\{ \text{reps of } \pi_1(U, u) \right\}.$$

$$\mathcal{L} \longmapsto \mathcal{L}_u$$

6. show that $\left\{ \begin{matrix} \text{reps} \\ \text{of } \pi_1(U,u) \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{local systems} \\ \text{on } \mathbb{C}^\times \end{matrix} \right\} \xrightarrow{i^* j_*} \text{Vect}$

sends a rep V of $\pi_1(U,u) \simeq \mathbb{Z}$ to its invariants $V^{\mathbb{Z}}$.

In particular, for a rank 1 local system $\mathcal{L}$, deduce that $i^* j_* \mathcal{L}$ is nonzero iff $\mathcal{L}$ is trivial.