# Derived Categories & Functors (1)

## I Motivation: $\mathcal{A}$ an abelian cat.

$A^\bullet, B^\bullet, C^\bullet$ chain cplxs / $\mathcal{A}$

$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ SES

Snake lemma ⇒ get LES

$$\to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to$$

weird obvious

**Derived cat**: put "weird" maps on equal footing with obvious ones.

**Defn.** $f: A^\bullet \to B^\bullet$ chain map is called qiso if $H^\cdot(A) \xrightarrow{\sim} H^\cdot(B)$.

**"Defn"** $D(\mathcal{A})$: take category of ch. cplxs $Ch(\mathcal{A})$ & "formally invert" all qisos.

variants: $D^+, D^-, D^b$.

$f: A^\bullet \to B^\bullet$ any chain map.

cone$(f)$: terms $A^{i+1} \oplus B^i$, diff. $\begin{bmatrix} -d_A & \\ f & d_B \end{bmatrix}$.

~~If f injective~~: $\boxed{\text{Distinguished tri.}}$:

$$A^\bullet \xrightarrow{f} B^\bullet \to \text{cone}(f) \to A[1]$$

If $f$ injective: ∃ qiso cone$(f) \downarrow$ cok$(f)$. ⇝ get LES.

If $f$ surj: ∃ qiso ker$(f)[1] \to$ cone$(f)$.

Key features

1) No more SES, only d.t.'s.
2) cone replaces ker and cok.
3) $D(\mathcal{A})$ NOT abelian in general
   – can't speak of "inj," "surj"

## II. Derived Functors

$F: \mathcal{A} \to \mathcal{B}$ additive functor.

Does it induce a functor

$$D(\mathcal{A}) \to D(\mathcal{B})\ ?$$

**Ans**: If $F$ exact, yes. In general, NO. problem: $F$ may not send qiso's to qiso's.

To fix: evaluate $F$ only on special complexes.

Assume $F$ is left exact.

**Fact**: If $f: A^\bullet \to B^\bullet$ is a qiso, & $A^\bullet$ & $B^\bullet$ are bdd below ch. cplxs w/ injective terms, then

$$F(f): F(A^\bullet) \to F(B^\bullet) \text{ is a qiso.}$$

**Defn**: $F$ left exact ⇝ rt derived functor

$$RF: D^+(\mathcal{A}) \to D^+(\mathcal{B}):$$

- For each $A^\bullet \in D^+(\mathcal{A})$, choose inj resol'n $A^\bullet \to I^\bullet$
  qiso to cplx w/ inj. terms

$$RF(A^\bullet) := F(I^\bullet).$$

- Well-defined up to can. isom
- univ. prop
- $\boxed{\text{takes d.t.'s to d.t.'s}}$

# Derived Categories & Functors ②

Classical derived functors:

$R^iF(X) = H^i(RF(X))$,

$\mathrm{Ext}^i(X,Y) = H^i R\mathrm{Hom}(X,Y)$.

## III. Examples

① R - comm ring.

$F = \mathrm{Hom}_R(-,R) : R\text{-mod}^{op} \to R\text{-mod}$.

- left exact (takes right exact to left exact.

$$\begin{tikzcd} RF = R\mathrm{Hom}_R(-,R) : D^+(R\text{-mod}^{op}) \arrow[d, equal] \arrow[r] & D^+(R\text{-mod}) \\ D^-(R\text{-mod})^{op} & \end{tikzcd}$$

- use inj. resolns in $R\text{-mod}^{op}$
  = proj. resolns in R-mod.

Specialize:

R a field: - usual duality.
- already exact.
- takes bdd to bdd.
- can iterate.

~~ev~~ nat'l map ev: $M \to RF(RF(M))$ M bdd.

- isom if $H^i(M)$ fin dim'l.

---

$R = \mathbb{Z}$: - takes bdd to bdd
- can iterate.

ev: $M \to RF(RF(M))$

- isom if $H^i(M)$ fin gen.

Work out for $M = \mathbb{Z}/n$.

Underived:

$\mathbb{Z}/n \to \mathrm{Hom}(\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}), \mathbb{Z})$
$\qquad\qquad\qquad \overset{\shortparallel}{0}$.

② $\mathbb{k}$ - a field.

$G = \mathbb{Z}/2$.

Look at G-reps / $\mathbb{k}$
= $\mathbb{k}[G]$ - modules.

$F : \mathbb{k}[G]\text{-mod} \to \mathbb{k}\text{-mod}$

$F(M) = M^G$ = G-invariants in M.

- F is left exact.
- If char $\mathbb{k} \neq 2$, F is exact.
- If char $\mathbb{k} = 2$:

$RF : D^+(\mathbb{k}[G]\text{-mod}) \to D^+(\mathbb{k}\text{-mod})$

~~Ex.~~ Compute $RF(\mathbb{k})$.

Use: $\mathbb{k}[G]$ is injective over itself.