

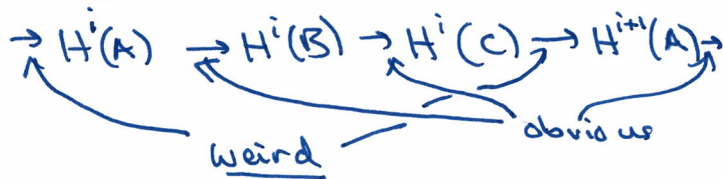
Derived Categories & Functors (1)

I Motivation: \mathcal{A} an abelian cat.

A^*, B^*, C^* chain cplx / \mathcal{A}

$0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$ SES

Snake lemma \Rightarrow get LES



Derived cat: put "weird" maps on equal footing with obvious ones.

Defn. $f: A^* \rightarrow B^*$ chain map is called qiso if $H^i(A) \cong H^i(B)$.

"Defn" $D(\mathcal{A})$: take category of ch. cplx $Ch(\mathcal{A})$ & "formally invert" all qisos.

variants: D^+, D^-, D^b .

$f: A^* \rightarrow B^*$ any chain map.

~~if f injective~~: Distinguished tri.:
 $A^* \xrightarrow{f} B^* \rightarrow \text{cone}(f) \rightarrow A^*[1]$

If f injective: \exists qiso $\text{cok}(f) \rightarrow \text{cone}(f)$. \Rightarrow get LES.

If f surj: \exists qiso $\text{ker}(f)[1] \rightarrow \text{cone}(f)$.

Key features

- 1) No more SES, only d.t.'s.
- 2) cone replaces ker and cok.
- 3) $D(\mathcal{A})$ NOT abelian in general
 - can't speak of "inj", "surj"

II. Derived Functors

$F: \mathcal{A} \rightarrow \mathcal{B}$ additive functor.

Does it induce a functor

$D(\mathcal{A}) \rightarrow D(\mathcal{B})$?

Ans: If F exact, yes.
 In general, No. Problem:
 F may not send qiso's to qiso's.

To fix: evaluate F only on special complexes.

Assume F is left exact.

Fact: If $f: A^* \rightarrow B^*$ is a qiso, & A^* & B^* are bdd below ch. cplx w/ injective terms, then

$F(f): F(A^*) \rightarrow F(B^*)$ is a qiso.

Defn: F left exact as rt derived functor

$RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$:

- For each $A^* \in D^+(\mathcal{A})$, choose inj resolu $A^* \rightarrow I^*$
 \uparrow
 qiso to cplx w/ inj. terms

$RF(A^*) := F(I^*)$.

- Well-defined up to can. isom
- univ. prop
- Takes d.t.'s to d.t.'s

Derived Categories & Functors (2)

Classical derived functors:

$$R^i F(x) = H^i(RF(x)),$$

$$\text{Ext}^i(x, y) = H^i R\text{Hom}(x, y).$$

(2) k - a field.

$$G = \mathbb{Z}/2.$$

Look at G -reps / k
 $= k[G]$ -modules.

$$F: k[G]\text{-mod} \rightarrow k\text{-mod}$$

$$F(M) = M^G = G\text{-invariants in } M.$$

F is left exact.

- If $\text{char } k \neq 2$, F is exact.

- If $\text{char } k = 2$:

$$RF: D^+(k[G]\text{-mod}) \rightarrow D^+(k\text{-mod})$$

~~Compute~~ Compute $RF(k)$.

Use: $k[G]$ is injective over itself.

III. Examples

(1) R - comm ring.

$$F = \text{Hom}_R(-, R): R\text{-mod}^{\text{op}} \rightarrow R\text{-mod}.$$

- left exact (takes right exact to left exact).

$$RF = R\text{Hom}_R(-, R): D^+(R\text{-mod}^{\text{op}}) \rightarrow D^+(R\text{-mod})$$

- use inj. resolves in $R\text{-mod}^{\text{op}}$
 - proj. resolves in $R\text{-mod}$.

Specialize:

R a field: - usual duality.
 - already exact.

- takes odd to odd.

- can iterate.

natl map

$$\text{ev}: M \rightarrow RF(RF(M)) \quad M \text{ odd.}$$

- isom if $H^i(M)$ fin dim.

$R = \mathbb{Z}$: - takes odd to odd

- can iterate.

$$\text{ev}: M \rightarrow RF(RF(M))$$

- isom if $H^i(M)$ fin gen.

Work out for $M = \mathbb{Z}/n$.

Underived:

$$\mathbb{Z}/n \rightarrow \text{Hom}(\text{Hom}(\mathbb{Z}/n, \mathbb{Z}), \mathbb{Z})$$

$\cong \mathbb{Z}/n$.