(Equivariant) derived categories of sheaves

1. To a topological space X, we associate $Sh(X)$, its abelian cat of sheaves.

Denote by $D(X) := D(Sh(X))$ the corresponding (bounded below) derived category.

2. Given $f : X \to Y$, we had an adjunction

$$f^* : Sh(Y) \rightleftarrows Sh(X) : f_*.$$

f^* played well with stalks hence was exact. So we get

$$f^* : D(Y) \longrightarrow D(X)$$

by just applying f^* termwise to a complex:

$$f^*\left(\cdots \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \right)$$
$$:= \quad \cdots \to f^*\mathcal{F}^0 \to f^*\mathcal{F}^1 \to f^*\mathcal{F}^2 \to \cdots$$

Note we still have $\left(f^*\mathcal{F}^\bullet\right)_x \simeq \mathcal{F}^\bullet_{f(x)}$, but these are now objects of the derived cat of vector spaces.

f_* was only left exact, so we will right derive it:

$$Rf_* : D(X) \to D(Y).$$

This is possible because:

<u>Lemma</u> $Sh(X)$ has enough injectives.

<u>pf.</u> For any $\mathcal{F}$, consider $\underset{\substack{\uparrow \\ \text{injective} \\ \text{map}}}{\mathcal{F} \to} \prod_{x \in X} \underset{\substack{\uparrow \\ \text{Injective} \\ \text{sheaf}}}{i_* i^* \mathcal{F}}$.

For formal reasons still have an adjunction

$$f^* : D(Y) \rightleftarrows D(X) : Rf_*,$$

but what does Rf_* "look like"?

<u>example</u> For $\pi : X \to pt$, we get from

$$\pi^* : \text{vect} \rightleftarrows sh(X) : \pi_*$$

on derived cat's

$$\pi^* : D(\text{vect}) \rightleftarrows D(X) : R\pi_*.$$

As $\pi_*(\mathcal{F}) = \text{Hom}(\mathbb{C}_X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

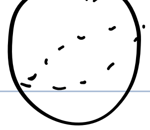
we get $R\pi_*(\mathcal{F}) = R\text{Hom}(\mathbb{C}_X, \mathcal{F}) =: R\Gamma(X, \mathcal{F})$.

$H^i R\pi_*(\mathcal{F}) = \text{Ext}^i(\mathbb{C}_X, \mathcal{F})$, $i \in \mathbb{Z}$.

<u>fact</u> For a reasonable topological space X,

$$\underset{\substack{\uparrow \\ \text{derived} \\ \text{pushforward} \\ \text{of constant} \\ \text{sheaf}}}{R\pi_*(\mathbb{C}_X)} \simeq \underset{\substack{\uparrow \\ \text{topological, i.e. singular,} \\ \text{cohomology of } X.}}{C^*(X, \mathbb{C})}$$

example: $X = \mathbb{P}^1$, get



$$\mathrm{Ext}^i(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = \begin{cases} \mathbb{C} & i = 0, 2 \\ 0 & \text{o.w.} \end{cases}$$

In general, Rf_* is computing derived sections on preimages of opens:

$$R\Gamma(U, Rf_* \mathcal{F}) \simeq R\Gamma(f^{-1}U, \mathcal{F}).$$

3. The 1st triangle:

Recall for $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$ a closed and open, we had

$$0 \to j_! j^* \to id \to i_* i^* \to 0.$$

As all functors appearing are exact, we get

$$j_! : D(U) \to D(X),$$

and a distinguished Δ:

$$j_! j^* \to id \to i_* i^* \to . \quad (1)$$

example $\{\infty\} \hookrightarrow S^n \hookleftarrow \mathbb{R}^n$, Get:

$$0 \to j_! \underline{\mathbb{C}}_{\mathbb{R}^n} \to \underline{\mathbb{C}}_{S^n} \to i_* \underline{\mathbb{C}}_{\{\infty\}} \to 0.$$

On global sections, get a d.t.

$$\begin{tikzcd}
R\Gamma(j_! \mathbb{C}_{\mathbb{R}^n}) \arrow[r] & R\Gamma(\mathbb{C}_{S^n}) \arrow[r] \arrow[d, phantom, "\wr"] & R\Gamma(i_* \mathbb{C}_{\{\infty\}}) \arrow[r, "+1"] \arrow[d, phantom, "\wr"] & {} \\
 & C^*(S^n) \arrow[r, "res"'] & C^*(\{\infty\}) &
\end{tikzcd}$$

hence a LES: yielding

$$R^i\Gamma(S^n, j_! \mathbb{C}_{\mathbb{R}^n}) = \begin{cases} \mathbb{C} & i = n \\ 0 & \text{o.w.} \end{cases}$$

This should remind us of

$$\underbrace{H^i_c(\mathbb{R}^n, \mathbb{C})}_{\text{compactly supported cohomology.}} = \begin{cases} \mathbb{C} & i = n \\ 0 & \text{o.w.} \end{cases}$$

More on this below.

4. The second triangle.

Recall we also had

$$0 \to i_* i^! \to id \to j_* j^* \quad \text{(not exact!!)}$$

$\uparrow$

sections
supported
on Z

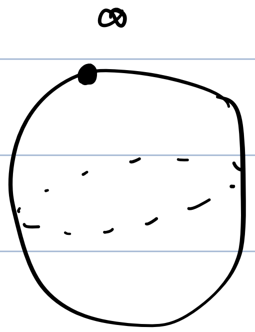
$i^!$ left exact $\Rightarrow$ $Ri^! : D(X) \to D(Z)$,

and we <u>do</u> get a d.t.

$$i_* \circ Ri^! \to id \to Rj_* \circ j^* \xrightarrow{+1} . \qquad (2)$$

Note the resemblance of (1) and (2), swapping j^* w/ $Ri^!$, $j_!$ w/ Rj_*, and direction of arrows.

<u>example</u> $\{\infty\} \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1$



Get
$$(Rj_* \mathbb{C}_U)_x = \begin{cases} \mathbb{C} & x \neq \infty, \\ C^*(\odot, \mathbb{C}) & x = \infty . \end{cases}$$
$$C^*(\odot, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}[-1]$$

The LES on cohomology associated to

$$i_* Ri^! \underline{\mathbb{C}}_{\mathbb{P}^1} \longrightarrow \underline{\mathbb{C}}_{\mathbb{P}^1} \longrightarrow Rj_* j^* \underline{\mathbb{C}}_{\mathbb{P}^1} \xrightarrow{+1} ,$$

yields

$$\begin{tikzcd}
R\Gamma(i_* Ri^! \underline{\mathbb{C}}_{\mathbb{P}^1}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H^*(\mathbb{P}^1) \arrow[r, "res"] \arrow[d, phantom, sloped, "\cong"] & H^*(\mathbb{A}^1) \arrow[r, "+1"] \arrow[d, phantom, sloped, "\cong"] & {} \\
Ri^! \underline{\mathbb{C}}_{\mathbb{P}^1} & \mathbb{C} \oplus \mathbb{C}[-2] & \mathbb{C} ,
\end{tikzcd}$$

so $$Ri^! \underline{\mathbb{C}}_{\mathbb{P}^1} \simeq \mathbb{C}[-2] \simeq H^2(\mathbb{P}^1).$$

In general, given a smooth point $x \in X$

$\uparrow$ WLOG connected

get $$Ri^! \underline{\mathbb{C}}_x \simeq \mathbb{C}[-2d_X]$$

$\uparrow$

complex dimension of X,

e.g. 1 for $\mathbb{P}^1$

5. Compactly supported cohomology

We know how to discuss $H^*(X)$ via sheaves, and similarly cohomology of pairs. What about compactly supported cohomology? Recall

$$\begin{tikzcd}
H^0_c(X) \arrow[r, hook] & H^0(X) \arrow[d, phantom, sloped, "\cong"] \\
\bigoplus\limits_{\substack{\gamma \in \pi_0(X) \\ \text{s.t. } X_\gamma \text{ is} \\ \underline{\text{compact}}}} \mathbb{C} \arrow[r, hook] & \bigoplus\limits_{\pi_0(X)} \mathbb{C} .
\end{tikzcd}$$

We can guess from here, and it works:

<u>def</u> Given $f : X \to Y$, have a functor

$$f_! : Sh(X) \to Sh(Y),$$

where

$$(f_! \mathcal{F})\underset{\substack{\uparrow \\ U \subset Y \\ \text{open}}}{(U)} = \left\{ \sigma \in \mathcal{F}(f^{-1}U) : \underset{\text{is } \underline{\text{proper}}}{\operatorname{supp}(\sigma) \to U} \right\}$$

In particular $0 \to f_! \mathcal{F} \to f_* \mathcal{F}$.

E.g., for $\pi : X \to pt$, we have

$$\pi_! \mathcal{F} =: R\Gamma_c(X, \mathcal{F}) = \left\{ \sigma \in \mathcal{F}(X) : \underset{\text{compact}}{\text{supp } \sigma \text{ is}} \right\}.$$

$f_!$ is straightforwardly left exact ⇝

$$Rf_! : D(X) \to D(Y). :$$

Naive hope works:

Fact For reasonable X, have

$$R\Gamma_c(X, \underline{\mathbb{C}}) \simeq C^*_c(X).$$

↑

compactly supported cohomology.

example $R\Gamma_c(\mathbb{C}^n, \underline{\mathbb{C}}) \simeq \mathbb{C}[-2d_X]$.

useful fact: (proper base change): Given a Cartesian square

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

, have $g^* \circ Rf_! \simeq R\tilde{f}_! \circ \tilde{g}^*$,

thm For reasonable X and Y,

$Rf_! : D(X) \to D(Y)$ admits a

right adjoint:

$$f^! : D(Y) \longrightarrow D(X),$$

$\uparrow$

Cool feature: not right derived functor of a right adjoint on abelian categories, i.e., truly a derived animal,

ex. For $j : U \to X$ an open embedding, $j_!$ is the functor from before (ext by zero), and $j^! = j^*$.

definition For $\pi : X \to pt$, we have

$$\omega_X := \pi^! \mathbb{C}.$$

$\uparrow$

dualizing sheaf

In particular: $R\Gamma(U, \omega_X) \simeq R\Gamma(U, \omega_U) \simeq R\mathrm{Hom}_{D(U)}(\mathbb{C}_U, \omega_U)$

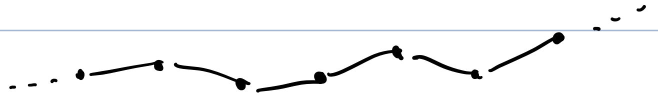
$$= R\mathrm{Hom}_{D(U)}(\mathbb{C}_U, \pi^! \mathbb{C}) \simeq R\mathrm{Hom}_{D(pt)}(\pi_! \mathbb{C}_U, \mathbb{C})$$

$$\simeq R\Gamma_c(U, \mathbb{C})^\vee,$$

I.e., dual to compactly supported cohomology

$\updownarrow$

chains w/ infinite support (cf Borel-Moore homology).



example X $\mathbb{R}$-manifold, have

$$\omega_X \simeq \text{orientation sheaf of } X;$$

$$x \in X: \quad i_x^* \omega_X \simeq \underbrace{\varinjlim_{U \ni x} H^*_c(U, \mathbb{C})^\vee}$$

line in degree $-\dim X$;
to trivialize need to pick orientation
on small ball about x.

cor If X is orientable,

$$\omega_X \underset{\substack{\uparrow \\ \text{pick orientation}}}{\simeq} \mathbb{C}_X[\dim X], \quad \text{and so}$$

$$\begin{tikzcd}
R\Gamma(X, \omega_X) \arrow[r, phantom, "="] \arrow[d, phantom, "\wr"] & R\Gamma(X, \mathbb{C}_X[\dim X]) \arrow[r, phantom, "\simeq"] & H^*(X, \mathbb{C})[\dim X] \\
R\Gamma_c(X, \mathbb{C})^\vee \arrow[urr, leftrightarrow, "\text{Poincaré duality!}"'] & &
\end{tikzcd}$$

6. local systems and constructible sheaves

Recall a sheaf $\mathcal{L}$ on X is called <u>locally constant</u> if $\forall x \in X$, $\exists\ U \ni x$ open such that $\mathcal{L}|_U$ is isomorphic (non-canonically!) to $\underline{\mathbb{C}}_U^{\oplus r}$ ← $r :=$ rank of $\mathcal{L}$.

Equivalent pictures: ① trivialize on a cover U_i, $i \in I$; specified by locally constant transition functions

$$g_{ij} : U_i \cap U_j \longrightarrow GL(r, \mathbb{C})$$

$$g_{ij}\, g_{jk} = g_{ik}. \qquad (H^1(X, \underline{GL(r,\mathbb{C})}))$$

② representations of fundamental group(oid) of X.

<u>example</u> $X = \mathbb{C}^\times$, $\pi_1(X) \simeq \mathbb{Z}$, so

$$\left\{ \begin{matrix} \text{rank } r \\ \text{local systems} \end{matrix} \right\} \Big/_{\text{iso}} \simeq GL_r(\mathbb{C}) \Big/^{\mathrm{Ad}}_{GL_r(\mathbb{C})}.$$

Orienting q: given a map $f: X \to Y$ of alg varieties, do f^* $f^!$ f_* $f_!$ preserve local systems?

f^*: yes! :o) $f^!\ f_*\ f_!$: no in general :o(

def The constructible derived category $D_c(X)$ is the full subcategory generated by $*$-pushforwards of local systems on locally closed subvarieties.

facts: $f^*\ f^!\ f_*\ f_!$ preserve constructibility, ω_X is constructible, and the dual

$$\mathbb{D}\mathcal{F} := \overset{\text{internal Hom w.r.t. } \otimes \text{ of sheaves}}{R\mathcal{H}om}(\mathcal{F}, \omega_X) \in D(X)$$

of a constructible sheaf is again constructible.

Moreover: $\mathbb{D}^2 \simeq id$,
on constructible sheaves
$\mathbb{D}Rf_*\mathbb{D} \simeq Rf_!$
$\mathbb{D}f^*\mathbb{D} \simeq f^!$

examples If X is smooth, and $\mathcal{L}$ is a local system, $\mathbb{D}\mathcal{L} \simeq \underset{\substack{\uparrow \\ \text{dual local system}}}{\mathcal{L}^\vee}[2\dim_{\mathbb{C}} X]$

$$\mathbb{D}\mathbb{D}\mathcal{L} \simeq (\mathcal{L}^{\vee\vee}[-2\dim_{\mathbb{C}} X])[2\dim_{\mathbb{C}} X] \simeq \mathcal{L}^{\vee\vee} \simeq \mathcal{L}.$$

__example__ If we apply $\mathbb{D}$ to a gluing triangle

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{+1} \qquad \mathcal{F} \in D_c(X),$$

we get:

$$\begin{tikzcd}
{} & \mathbb{D}\, j_! j^* \mathcal{F} \arrow[l, "+1"'] \arrow[d, phantom, "\wr"] & \mathbb{D}\mathcal{F} \arrow[l] & \mathbb{D}\, i_* i^* \mathcal{F} \arrow[l] \arrow[d, phantom, "\wr", "(i_* \underset{\text{closed}}{\simeq} i_!)"'] \\
{} & Rj_* \mathbb{D} j^* \mathcal{F} \arrow[d, phantom, "\wr", "(j^* \underset{\text{open}}{\simeq} j^!)"'] & & i_* \mathbb{D} i^* \mathcal{F} \arrow[d, phantom, "\wr"] \\
{} & Rj_* j^* \mathbb{D}\mathcal{F} & & i_* Ri^! \mathcal{F}
\end{tikzcd}$$

i.e. the triangle $i_* Ri^! \to id \to Rj_* j^* \xrightarrow{+1}$ for $\mathbb{D}\mathcal{F}$.

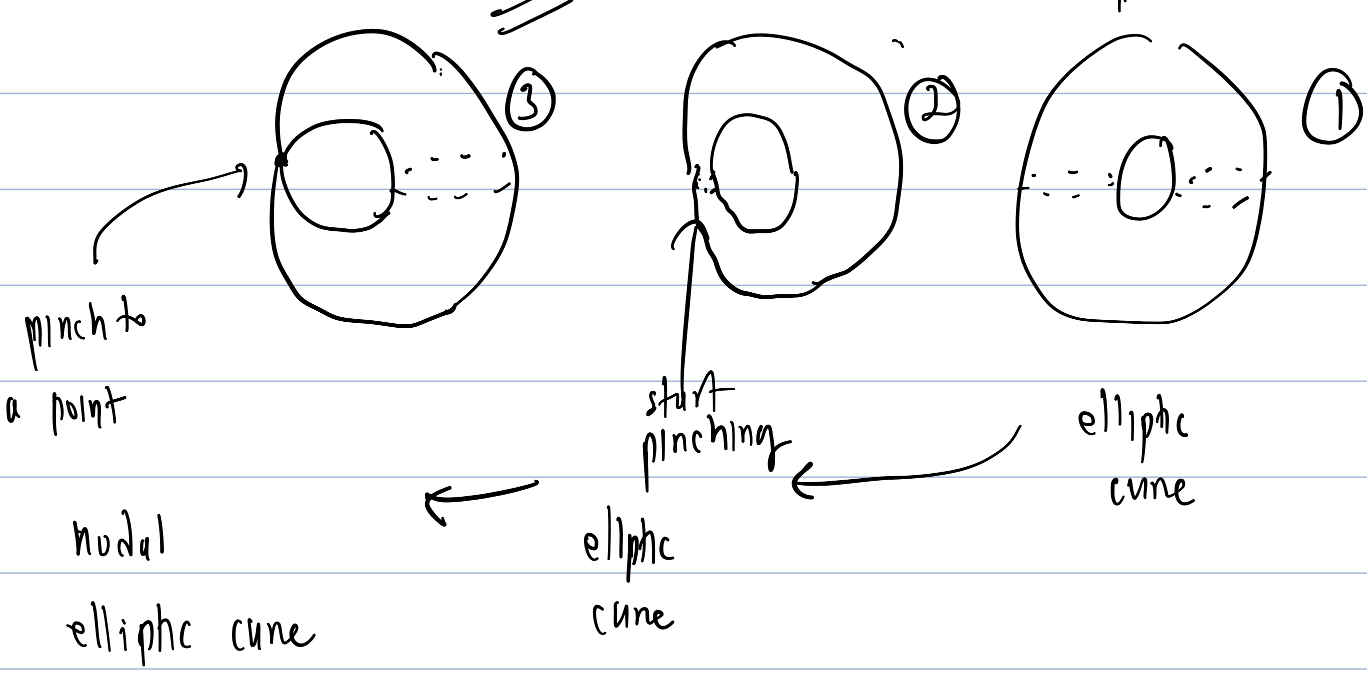
__example__ (Kunneth formula) Given
$$\begin{tikzcd} X & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] & Y \\ x & (x,y) \arrow[l] \arrow[r] & y \end{tikzcd},$$

and sheaves $\mathcal{F} \in D(X)$, $\mathcal{G} \in D(Y)$, we set:

$$\mathcal{F} \boxtimes \mathcal{G} := \pi_X^* \mathcal{F} \otimes \pi_Y^* \mathcal{G}.$$

Then if $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1, \mathcal{G}_2$ are constructible, $R\mathcal{H}om(\mathcal{F}_1 \boxtimes \mathcal{G}_1, \mathcal{F}_2 \boxtimes \mathcal{G}_2) \simeq R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) \boxtimes R\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$.

exercise   Suppose X is a nodal elliptic curve:



⓪ Writing $j : U \to X$ for the smooth locus of  ,
compute the stalks of $Rj_* \underline{\mathbb{C}}_U$ X

① Use your answer to ⓪ to compute
the !-stalks of $\underline{\mathbb{C}}_X$ at all points $x \in X$,
i.e. $i_x^! \underline{\mathbb{C}}_X$, $\forall x \in X$,

② Use your answer to ① to compute
the *-stalks of ω_X.

③ Deduce $\omega_X \not\cong \underline{\mathbb{C}}_X$, but is still
constructible.

Equivariant derived categories :

suppose $\underset{\substack{\uparrow \\ \text{alg group}}}{G} \curvearrowright \underset{\substack{\uparrow \\ \text{variety}}}{X}$. Want a good notion of

constructible sheaves on X/G ← space of G-orbits, which

remembers stabilizers ⇝

$$\begin{tikzcd} D_c(X/G) \arrow[r, phantom, "\sim"] \arrow[d, hook] & D_c^G(X), \\ D(X/G) \arrow[r, phantom, "\sim"] & D^G(X) \end{tikzcd}$$

example want for $X = pt$ that

$$\mathrm{RHom}_{D(\cdot/G)}(\mathbb{C}, \mathbb{C}) \simeq H^*(BG, \mathbb{C})$$

$$\left(\begin{aligned} &\simeq \mathbb{C}[c], \ |c| = 2, \ \text{for } G = \mathbb{G}_m, \\ &\simeq \mathrm{Sym}\, \mathfrak{t}^*[-2], \ \text{for } G = T \text{ torus}, \\ &\simeq (\mathrm{Sym}\, \mathfrak{t}^*[-2])^W, \ \text{for general } G, \\ &\qquad \text{w/ max torus } T \\ &\qquad \text{and weyl group } W. \end{aligned} \right)$$

example want that for

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \circlearrowleft & \circlearrowleft \\ H \arrow[r] & G \end{tikzcd} \qquad (\rightsquigarrow X/H \to Y/G)$$

should have

$$\begin{tikzcd} Rf_!, Rf_* : & D(X/H) \arrow[r] & D(Y/G) \\ & D_c(X/H) \arrow[r] \arrow[u, hook] & D_c(Y/G) \arrow[u, hook] \end{tikzcd}$$

$$\begin{tikzcd} f^!, f^* : & D(Y/G) \arrow[r] & D(X/H). \\ & D_c(Y/G) \arrow[r] \arrow[u, hook] & D_c(X/H) \arrow[u, hook] \end{tikzcd}$$

and have $\mathbb{D} : D_c(X/H) \to D_c(X/H)$, $D_c(Y/G) \to D_c(Y/G)$

satisfying usual identities.

fact $D_c(X/G)$ & $D(X/G)$ exist, and $D(X/G)$ contains the abelian category of equivariant sheaves;

definition A G-equivariant sheaf on X is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F} \in Sh(X)$ and α is an isomorphism:

$$\alpha : act^* \mathcal{F} \simeq pr^* \mathcal{F},$$

where $act : G \times X \to X$, $(g, x) \mapsto gx$, $pr : G \times X \to X$, $(g, x) \to x$,

satisfying the following cocycle compatibility:

$$\alpha|_{(g,x)} \approx \mathcal{F}_{gx} \simeq \mathcal{F}_x$$

want

$$\begin{tikzcd}
\mathcal{F}_{ghx} \arrow[rr, "\alpha|_{(gh,x)}", "\sim"'] \arrow[dr, "\sim"', "\alpha|_{(g,hx)}"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{F}_x \\
& \mathcal{F}_{hx} \arrow[ur, "\sim", "\alpha|_{(h,x)}"'] &
\end{tikzcd}$$

i.e, identity on $G \times G \times X$;

example: If $X = pt$, an object of $Sh^G(pt) \leftrightarrow$ rep of $\pi_0(G)$.

example If $X = G/H$, an object of $Sh^G(X) \leftrightarrow$ rep of $\pi_0(H)$.