Perverse Sheaves ①

I. Constructible Complexes

(Maybe skip or shorten depending on preceding lectures)

X - variety

$(X_s)_{s\in\mathcal{S}}$ stratification i.e.

- each X_s smooth, loc. closed, conn.
- $\overline{X_s}$ is a union of strata
- $\mathcal{S}$ is finite.

Recall: $\mathbb{k}$ - field.

local systems on X = loc const. sheaf of $\mathbb{k}$-vec sp $\longleftrightarrow$ rep. of $\pi_1(X)$ on $\mathbb{k}$-vec. sp.

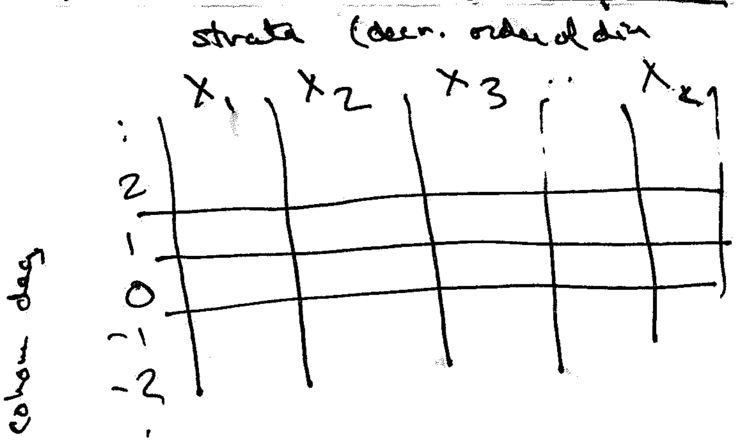
A sheaf of $\mathbb{k}$-vec sp on X is **constructible** w.r.t. $(X_s)_{s\in\mathcal{S}}$ if each $\mathcal{F}|_{X_s}$ is a loc sys of fin. rank.

fin dim'l rep. of π_1.

A complex $\mathcal{F}\in D^b(X,\mathbb{k})$ is **constr.** if each $\mathcal{H}^i(\mathcal{F})$ is constr.

only fin. many.

To write down: **table of stalks**



entries: $\mathcal{H}^i(\mathcal{F})|_{X_j}$ as rep of $\pi_1(X_j)$.

If $\pi_1(X_j)$ triv: write $\mathbb{k}^n$.

Verdier duality

$$\mathbb{D} = R\mathcal{H}om(\ \ ,\omega_X).$$

- $\mathbb{D}\circ\mathbb{D} = \mathrm{id}$.
- If X smooth,

$$\underset{\text{loc sys}}{\mathbb{D}(\mathcal{L})} = \underset{\text{dual loc sys.}}{\mathcal{L}^\vee}[2\dim X]$$

Mild assump on $(X_s)_{s\in\mathcal{S}}$ $\Rightarrow$ ($\mathcal{F}$ constr. $\Leftrightarrow$ $\mathbb{D}\mathcal{F}$ constr.).

II. Perverse sheaves

Defn. $\mathcal{F}\in D^b_{\mathcal{S}}(X,\mathbb{k})$ is **perverse** if

- table of stalks is "below the diagonal"

$$\mathcal{H}^i(\mathcal{F})|_{X_s} = 0 \quad \forall i > -\dim X_s.$$

- same for $\mathbb{D}\mathcal{F}$.

Examples: $\mathbb{P}^1$, pt $\cup\ \mathbb{A}^1$

$j_!\mathbb{k}[1]$, $j_*\mathbb{k}[1]$, $\underline{\mathbb{k}}[1]$, skyscraper.

- $\mathcal{N}$ for SL_2. $\mathbb{k}=\mathbb{C}$, $\mathbb{F}_2$.
- rank 1 3×3 matrices.

Perverse Sheaves ②

Thm Perv(X) is an abelian cat.

Pf Use "t-structures". Main idea: $\phi : \mathcal{F} \to \mathcal{G}$ in Perv(X).

$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \to \underset{\text{cone}}{\mathcal{K}} \to$ ← below diagonal, but $\mathbb{D}\mathcal{K}$ is not.

Hands-on constr: $\exists$

$\mathcal{A}[1] \to \mathcal{K} \to \mathcal{B} \to$ $\quad \mathcal{A}, \mathcal{B}$ Perv.

$\leadsto$ $\mathcal{A} = \ker \phi$, $\mathcal{B} = \operatorname{cok} \phi$.

$\leadsto$ 4 term exact seq.

$0 \to \mathcal{A} \to \mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$. □

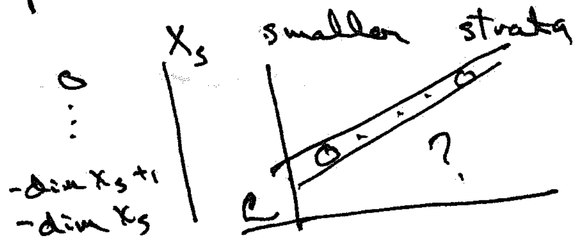
Generalize "perverse cohomology"

$\mathcal{K} \leadsto {}^p\mathcal{H}^i(\mathcal{K}) \in$ Perv.

$\mathcal{A} = {}^p\mathcal{H}^{-1}(\mathcal{K})$, $\quad \mathcal{B} = {}^p\mathcal{H}^0(\mathcal{K})$.

Any d.t. $\leadsto$ LES of perv. sh.

Thm. $\forall$ strat. X_s, irred loc sys $\mathcal{L}$, $\exists!$ perv. sh. $IC(X_s, \mathcal{L})$ s.t.



$+ \ \mathbb{D}$.

This is simple & every simple looks like this.

Every perv. sh has a comp. series.

Examples on $\mathbb{P}^1$.

Some facts:

Thm. $f : X \to Y$ smooth mor. of rel. dim d.

Then $f^*[d] = f^![d]$ ~~$\mathcal{B}$~~

- sends perv. to perv.
- faithful on perv. sh.
- If conn. fibers: fully faithful

Thm. $j : Y \hookrightarrow X$ inclusion of a loc. closed subvariety

If j is affine (e.g. Y affine)

$j_!$, j_* send Perv to perv.

Example: $\mathbb{P}^1$

Non-example: $\mathcal{N}$ for SL_2.

Thm. $f : X \to Y$ proper, X smooth, Y strat., ~~[illegible]~~ irred.

$y \in Y_t \Rightarrow \dim f^{-1}(y) \le \frac{1}{2} \operatorname{codim} Y_t$.

("semismall").

Then $f_* \Bbbk[\dim X]$ is perv.