Tannakian formalism

We're trying to show that we have an equivalence of (symmetric) monoidal categories

$$\mathrm{Perv}\left(G_0 \backslash G_F / G_0\right) \overset{\otimes}{\simeq} \mathrm{Rep}(\check{G})^{f.d.}$$

f.d. ← finite dim reps

How can one recognize $\mathrm{Rep}(H)$ as a (symmetric) monoidal category, for H an algebraic group?

Well, first note that $\mathrm{Rep}(H)^{f.d.}$ comes w/ a forgetful functor to $\mathrm{Vect}^{f.d.}$:

$$\mathrm{Oblv}: \mathrm{Rep}(H)^{f.d.} \xrightarrow{\otimes} \mathrm{Vect}^{f.d.}$$

This is conservative for silly reasons.
(Is it unique - a priori unclear)
The great thing is that's it!

<u>thm</u> (Tannaka duality) suppose $\mathcal{C}$ is a k-linear symmetric monoidal category equipped with a conservative functor

field ↙ (k)

$$F: \mathcal{C} \xrightarrow{\otimes} \mathrm{Vect}_k^{f.d.}$$

Then $\exists !$ algebraic group H and a $!$ $\otimes$-equivalence fitting into:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\wr"'] \arrow[dr, "\otimes"] & \\ \mathrm{Rep}(H)^{f.d.} \arrow[r, "\mathrm{Obl}"'] & \mathrm{Vect}^{f.d.} \end{tikzcd}$$

can be skipped!

Idea of proof: we know the animal controlling $\mathrm{Rep}(H)^{f.d.}$ is $\mathcal{O}_H$, but it's not yet visible. How to add?

Take <u>Ind-completion</u>:

Given a category $\mathcal{A}$, one can make another cat $\mathrm{Ind}(\mathcal{A})$, w/ objects direct limits of objects of $\mathcal{A}$:

$(a_i)_{i \in I}$ ← some directed set $\qquad a_i \xrightarrow{\phi_{ji}} a_j \qquad i \le j,$

$$\phi_{kj}\phi_{ji} = \phi_{ki} \qquad i \le j \le k.$$

and Hom's compatible maps between steps:

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{A})}\left(\text{"}\varinjlim\text{"}(a_i)_{i \in I}, \text{"}\varinjlim\text{"}(\alpha_j)_{j \in J} \right) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}(a_i, \alpha_j).$$

<u>example.</u> $\mathrm{Ind}(\mathrm{Vect}^{f.d.}) \simeq \mathrm{Vect}$ ← all v.s., not necc f.d.

$\mathrm{Ind}(\mathrm{Rep}(H)^{f.d.}) \simeq \mathrm{Rep}(H)$ ← all comodules, not necc f.d.

$\mathcal{O}_H$ ← which makes us happy

So, we still have conservative

$$F : \mathrm{Ind}(\mathcal{C}) \xrightarrow{\otimes} \mathrm{Vect}$$

For nonsense reasons, namely that F commutes w/ arbitrary $\oplus$'s, this admits a right adjoint:

$$\mathrm{Ind}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Vect}. \qquad \left(\mathrm{Rep}(H) \underset{\mathrm{Coind}}{\overset{\mathrm{Oblv}}{\rightleftarrows}} \mathrm{Vect} \right)$$

By conservativity, $\mathrm{Ind}(\mathcal{C})$ must be comodules for the comonad FG:

$$\begin{tikzcd} \mathrm{Ind}(\mathcal{C}) \arrow[rr, "\sim"] \arrow[dr, "F"'] & & FG\text{-comod} \arrow[dl, "\mathrm{oblv}"] \\ & \mathrm{Vect} & \end{tikzcd}$$

(this is Barr-Beck theorem). But, since F & G commute w/ $\oplus$'s, this is just $- \otimes FG(k)$, which acquires a coalgebra structure:

$$FG(k) \xrightarrow{F(1 \to GF)G} FGFG(k) \cong FG(k) \otimes FG(k).$$

I.e., we showed:

Lemma Any k-linear category $\mathcal{C}$ equipped w/ an adjunction

$$F : \mathcal{C} \rightleftarrows \mathrm{Vect} : G$$

where G commutes w/ $\oplus$'s is of the form:

$$\mathcal{O}\text{-comod} \underset{C\otimes -}{\overset{\mathrm{oblv}}{\rightleftarrows}} \mathrm{Vect}$$

for a ! coalgebra $\mathcal{O}$ (namely $FG(k)$).

From here, we have also a $\otimes$ on our $\mathcal{C}$:

$$\begin{tikzcd} \mathcal{O}\text{-comod} \times \mathcal{O}\text{-comod} \arrow[r] \arrow[d] & \mathcal{O}\text{-comod} \arrow[d] \\ \mathrm{Vect} \times \mathrm{Vect} \arrow[r] & \mathrm{Vect} \end{tikzcd};$$

this gives us an algebra structure on $\mathcal{O}$.

$$\mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O},$$ ← map of also coalgebras

i.e. $\mathcal{O}$ is a bialgebra; i.e.

Lemma Any monoidal cat $\mathcal{C}$ equipped w/ an adjunction

$$\mathcal{C} \underset{G}{\overset{F_{\otimes}}{\rightleftarrows}} \mathrm{Vect},$$

with F cont. G commuting w/ $\oplus$'s, is of the form

$$\mathcal{O}\text{-mod} \underset{G}{\overset{F_{\otimes}}{\rightleftarrows}} \mathrm{Vect} \qquad \text{for a } ! \text{ bi-algebra } \mathcal{O}.$$

Since $\otimes$ was symmetric monoidal, this forces

$$\mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$$

to be invariant under swap, i.e. $\mathcal{O}$ commutative algebra,

and having enough f.d. objects (dualizable) gives the antipode / inversion map on $\mathcal{O}$, i.e. $\mathcal{O}$ is a commutative Hopf algebra, aka $\mathcal{O}_H$ for some group variety* H.

* here using char $k = 0$, otherwise group scheme.

properties: Q: How do we recover $G(k)$?

A: Any $g \in G(k)$ yields a map of underlying v.s.'s

$$V \xrightarrow{g_V} V \qquad \forall \text{ rep's } V \text{ of } G.$$

which is functorial and plays well w/ $\otimes$:

$$g_{V\otimes W} = g_V \otimes g_W.$$

I.e. obtain a map

$$G(k) \longrightarrow \mathrm{Aut}^{\otimes}(\mathrm{Oblv})$$

aut of fiber functor.

This is a bijection! I.e. $G(k) \simeq \mathrm{Aut}^{\otimes}(\mathrm{Oblv})$.*

(pf. $\mathrm{Aut}(\mathrm{Oblv}) \approx$ automorphism of $\mathcal{O}_G$ as a left comodule over itself,

$\mathrm{Aut}^{\otimes}(\mathrm{Oblv}) \approx$ aut of $\mathcal{O}_G$ as a left comodule + as an algebra

$\leftrightarrow$ right translation by some $g \in G(k)$)

Q: How to recover maps of groups?

A: Given $H \to G$, get a map

* can upgrade to recover alg. group, not just set of k-pts, too.

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[rr, "\mathrm{Res}", "\otimes"'] \arrow[dr, "\mathrm{Oblv}"'] & & \mathrm{Rep}(H) \arrow[dl, "\mathrm{Oblv}"] \\ & \mathrm{Vect} & \end{tikzcd}$$

Conversely, any $\otimes$-functor in a diagram

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[rr, "\otimes"'] \arrow[dr, "\mathrm{Oblv}"'] & & \mathrm{Rep}(H) \arrow[dl, "\mathrm{Oblv}"] \\ & \mathrm{Vect} & \end{tikzcd}$$

arises from a ! map of alg groups $H \to G$.

(pf. $\begin{tikzcd} \mathrm{Rep}(G) \arrow[rr] \arrow[dr, "\mathrm{Oblv}"'] & & \mathrm{Rep}(H) \arrow[dl, "\mathrm{Oblv}"] \\ & {} & \end{tikzcd}$ $\longleftrightarrow$ map of coalgebras $\mathcal{O}_G \to \mathcal{O}_H$

Vect

$$\mathrm{Rep}(G) \xrightarrow{\otimes} \mathrm{Rep}(H)$$

$$\Updownarrow$$

map of bialgebras

$$\mathcal{O}_G \longrightarrow \mathcal{O}_H.$$

<u>exercise</u> Show that $\mathfrak{h}$ = Lie algebra of H, is identified w/ infinitesimal automorphisms of the forgetful functor, i.e.

natural transformations $\xi_V : \mathrm{Oblv}\, V \to \mathrm{Oblv}\, V$ $\forall V \in \mathrm{Rep}(H)$ satisfying

$$\xi_{V \otimes W} = \xi_V \otimes \mathrm{id}_W + \mathrm{id}_W \otimes \xi_W \quad \forall\, V, W \in \mathrm{Rep}(H).$$

(Hint: identify these ξ w/ derivations* of $\mathcal{O}$ which are maps of left $\mathcal{O}$ comodules, and check or believe that restricting those to the identity tangent space yields

$$\mathrm{Der}(\mathcal{O}_H)^H \xrightarrow{\sim} \mathfrak{h}\ )$$

* A derivation of a commutative algebra A is a k-linear map $\partial : A \to A$: $\partial(ab) = a\partial(b) + \partial(a)b$