

# Tannakian formalism

We're trying to show that we have an equivalence of <sup>(symmetric)</sup> monoidal categories

$$\text{Per}(\mathbb{G}_0 \setminus \mathbb{G}_F / \mathbb{G}_0) \stackrel{\otimes}{\simeq} \text{Rep}(\mathbb{G})^{\text{f.d.}} \leftarrow \begin{array}{l} \text{finite dim} \\ \text{reps} \end{array}$$

How can one recognize  $\text{Rep}(H)$  as a (symmetric) monoidal category, for  $H$  an algebraic group?

well, first note that  $\text{Rep}(H)^{\text{f.d.}}$  comes w/ a forgetful functor to  $\text{Vect}^{\text{f.d.}}$ :

$$\text{Oblv}: \text{Rep}(H)^{\text{f.d.}} \xrightarrow{\otimes} \text{Vect}^{\text{f.d.}}$$

This is conservative for silly reasons.

(Is it unique - a priori unclear)

The great thing is that's it!

thm (Tannaka duality) suppose  $\mathcal{C}$  is a  $k$ -linear symmetric monoidal category equipped with a conservative functor

$$F: \mathcal{C} \xrightarrow{\otimes} \text{Vect}_k^{\text{f.d.}}$$

Then  $\exists!$  algebraic group  $H$  and  $\alpha!$

$\otimes$ -equivalence fitting into:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\otimes} & \text{Vect}^{\text{f.d.}} \\ \downarrow & & \uparrow \\ \text{Rep}(H)^{\text{f.d.}} & \xrightarrow{\text{oblv}} & \end{array}$$

can be skipped!

Idea of proof: we know the animal controlling  $\text{Rep}(H)^{\text{f.d.}}$  is  $\mathcal{O}_H$ , but it's not yet visible. How to add?

Take Ind-completion:

Given a category  $\mathcal{A}$ , one can make another cat  $\text{Ind}(\mathcal{A})$ , w/ objects direct limits of objects of  $\mathcal{A}$ :

$$(a_i)_{i \in I} \leftarrow \begin{array}{ccc} a_i & \xrightarrow{\phi_{ji}} & a_j \\ & & i \leq j \end{array}$$

some directed set

$$\phi_{kj} \phi_{ji} = \phi_{ki} \quad i \leq j \leq k.$$

and Hom's compatible maps between steps:

$$\text{Hom}_{\text{Ind}(\mathcal{A})} \left( \varinjlim (a_i)_{i \in I}, \varinjlim (a_j)_{j \in J} \right) \cong \varprojlim_i \varinjlim_j \text{Hom}(a_i, a_j).$$

example.  $\text{Ind}(\text{Vect}^{\text{f.d.}}) \cong \text{Vect} \leftarrow$  all v.s., not necc f.d.  
 $\text{Ind}(\text{Rep}(H)^{\text{f.d.}}) \cong \text{Rep}(H) \leftarrow$  all comodules, not necc f.d.

$\mathbb{0}_H \leftarrow$  which makes us happy

So, we still have conservative

$$F: \text{Ind}(e) \xrightarrow{\otimes} \text{Vect}$$

For nonsense reasons, namely that  $F$  commutes w/  
arbitrary  $\otimes$ 's, this admits a right adjoint:

$$\text{Ind}(e) \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \text{Vect}. \quad \left( \begin{array}{c} \text{oblv} \\ \text{Rep}(H) \xrightleftharpoons{\quad} \text{Vect} \\ \text{Coind} \end{array} \right)$$

By conservativity,  $\text{Ind}(e)$  must be comodules for  
the monad  $FG$ :

$$\begin{array}{ccc} \text{Ind}(e) & \xrightarrow{\sim} & FG\text{-comod} \\ & \searrow F & \swarrow \text{oblv} \\ & & \text{Vect} \end{array}$$

(this is Barr-Beck theorem). But, since  $F$  &  $G$  commute w/ $\otimes$ 's,  
this is just  $- \otimes FG(k)$ , which acquires a  
coalgebra structure:

$$FG(k) \xrightarrow{F(1 \rightarrow GF)G} FG(FG(k)) \cong FG(k) \otimes FG(k).$$

I.e., we showed:

$k$ -linear category

lemma Any  $\mathcal{C}$  equipped w/ an adjunction

$$F: \mathcal{C} \rightleftarrows \text{Vect} : G$$

where  $G$  commutes w/  $\otimes$ 's is of the form:

$$\begin{array}{ccc} \mathcal{O}\text{-mod} & \xrightarrow{\text{oblv}} & \text{Vect} \\ & \xleftarrow{\text{Co-}} & \end{array}$$

for a ! coalgebra  $\mathcal{O}$  (namely  $FG(k)$ ).

From here, we have also a  $\otimes$  on our  $\mathcal{C}$ :

$$\begin{array}{ccc} \mathcal{O}\text{-mod} \times \mathcal{O}\text{-mod} & \longrightarrow & \mathcal{O}\text{-mod} \\ \downarrow & & \downarrow \\ \text{Vect} \times \text{Vect} & \longrightarrow & \text{Vect} \end{array} ;$$

this gives us an algebra structure on  $\mathcal{O}$ .

$$\mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$$

← map of <sup>also</sup> coalgebras

i.e.  $\mathcal{O}$  is a bialgebra, i.e.

lemma Any monoidal cat  $\mathcal{C}$  equipped w/ an adjunction

$$e \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \text{Vect},$$

with  $F$  const.  $G$  commuting w/  $\otimes$ 's, is of the form

$$\mathcal{O}\text{-mod} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \text{Vect} \quad \text{for a ! bi-algebra } \mathcal{O}.$$

Since  $\otimes$  was symmetric monoidal, this forces

$$\mathcal{O} \otimes \mathcal{O} \longrightarrow \mathcal{O}$$

to be invariant under swap, i.e.  $\mathcal{O}$  commutative algebra,

and having enough f.d. objects (dualizable) gives

the antipode / inversion map on  $\mathcal{O}$ , i.e.

$\mathcal{O}$  is a commutative Hopf algebra, aka

$\mathcal{O}_H$  for some group variety\*  $H$ .

\* here using char  $k = 0$ , otherwise group scheme.

properties: Q: How do we recover  $G(k)$ ?

A: Any  $g \in G(k)$  yields a map  
of underlying vs.'s

$$V \xrightarrow{g_V} V \quad V \text{ rep's } V \text{ of } G.$$

which is functorial and plays  
well w/  $\otimes$ :

$$g_{V \otimes W} = g_V \otimes g_W.$$

I.e. obtain a map

$$G(k) \longrightarrow \text{Aut}^{\otimes}(\text{oblv})$$

↑  
aut of fiber functor.

This is a bijection! I.e.  $G(k) \cong \text{Aut}^{\otimes}(\text{oblv})^*$

( pf.

$\text{Aut}(\text{oblv}) \cong$  automorphism of  $\mathcal{O}_G$  as  
a left module over itself,

$\text{Aut}^{\otimes}(\text{oblv}) \cong$  aut of  $\mathcal{O}_G$  as

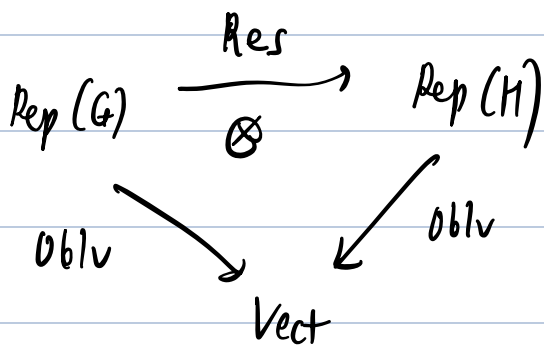
a left module + as an algebra

$\Leftrightarrow$  right translation by some  
 $g \in G(k)$

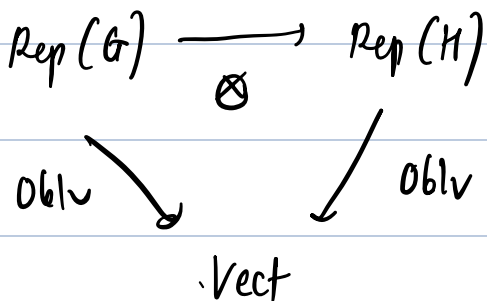
Q: How to recover maps of groups?

A: Given  $H \rightarrow G$ , get a map

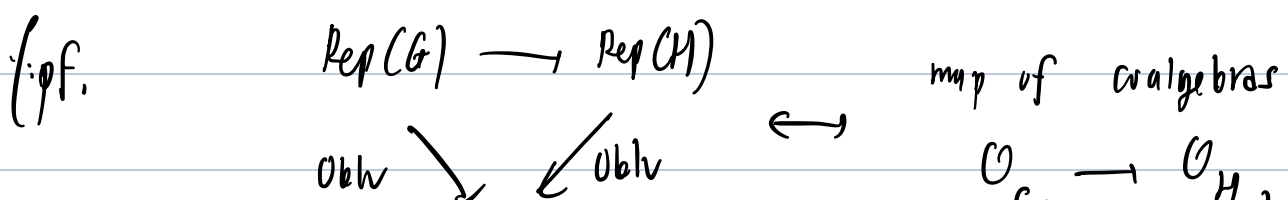
\* can upgrade to recover alg. group, not just set of  $k$ -pts, too.



Conversely, any  $\otimes$ -functor in a diagram



arises from a ! map of alg groups  $H \rightarrow G$ .



Vect

$$\text{Rep}(G) \xrightarrow{\otimes} \text{Rep}(H)$$

$\downarrow$   
map of bialgebras

$$\mathcal{O}_G \rightarrow \mathcal{O}_H.$$

exercise

Show that  $\mathfrak{h} = \text{Lie algebra of } H$ ,  
is identified w/ infinitesimal automorphisms  
of the forgetful functor, i.e.

natural transformations  $\xi_V : \text{ob } \mathcal{M} \rightarrow \text{ob } \mathcal{V} \quad \forall V \in \text{Rep}(H)$

satisfying

$$\xi_{V \otimes W} = \xi_V \otimes \text{id}_W + \text{id}_V \otimes \xi_W \quad \forall V, W \in \text{Rep}(H).$$

(Hint: identify these  $\xi_W$  w/ derivations\* of

$\mathcal{O}$  which are maps of left

$\mathcal{O}$  comodules, and check or

believe that restricting those to the

identity tangent space yields

$$\text{Der}(\mathcal{O}_H)^H \xrightarrow{\sim} \mathfrak{h}$$

\* A derivation of a commutative algebra  $A$  is a  $k$ -linear map  $d: A \rightarrow A$  :  $d(ab) = a d(b) + d(a)b$