REMINDER ON CATEGORIES AND FUNCTORS

1. Definitions

1.1. Categories.

Definition 1.1. A *category* $\mathscr{A}$ is the datum of:

- a set $\mathrm{Obj}(\mathscr{A})$ of "objects" of $\mathscr{A}$;
- for any $X, Y \in \mathrm{Obj}(\mathscr{A})$, a set $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ of "morphisms from X to Y;"
- for any $X, Y, Z \in \mathrm{Obj}(\mathscr{A})$, a map
$$\begin{cases} \mathrm{Hom}_{\mathscr{A}}(Y, Z) \times \mathrm{Hom}_{\mathscr{A}}(X, Y) & \to & \mathrm{Hom}_{\mathscr{A}}(X, Z) \\ (g, f) & \mapsto & g \circ f \end{cases}$$
defining "compositions of morphisms"

which satisfy the following conditions:

- (existence of identities) for any $X \in \mathrm{Obj}(\mathscr{A})$ there exists an element $\mathrm{id}_X \in \mathrm{Hom}_{\mathscr{A}}(X, X)$ such that for any $Y \in \mathrm{Obj}(\mathscr{A})$ and any $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, resp. $g \in \mathrm{Hom}_{\mathscr{A}}(Y, X)$, we have $f \circ \mathrm{id}_X = f$, resp. $\mathrm{id}_X \circ g = g$;
- (associativity) for any $X, Y, Z, W \in \mathrm{Obj}(\mathscr{A})$ and any $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, $g \in \mathrm{Hom}_{\mathscr{A}}(Y, Z)$, $h \in \mathrm{Hom}_{\mathscr{A}}(Z, W)$ we have
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Given $X, Y \in \mathrm{Obj}(\mathscr{A})$, an element $f \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$ will usually be denoted $f : X \to Y$.

Definition 1.2. A morphism $f : X \to Y$ is called an *isomorphism* if there exists a morphism $f^{-1} : Y \to X$ such that $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$.

Definition 1.3. If $\mathscr{A}$ is a category, the *opposite category* is the category $\mathscr{A}^{\mathrm{op}}$ with:

- objects $\mathrm{Obj}(\mathscr{A})$;
- for $X, Y \in \mathrm{Obj}(\mathscr{A}^{\mathrm{op}}) = \mathrm{Obj}(\mathscr{A})$, morphisms given by $\mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathscr{A}}(Y, X)$;
- for any $X, Y, Z \in \mathrm{Obj}(\mathscr{A}^{\mathrm{op}}) = \mathrm{Obj}(\mathscr{A})$, the composition map
$$\mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(Y, Z) \times \mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(X, Y) \to \mathrm{Hom}_{\mathscr{A}^{\mathrm{op}}}(X, Z)$$
given by
$$\begin{cases} \mathrm{Hom}_{\mathscr{A}}(Z, Y) \times \mathrm{Hom}_{\mathscr{A}}(Y, X) & \to & \mathrm{Hom}_{\mathscr{A}}(Z, X) \\ (g, f) & \mapsto & f \circ g \end{cases}.$$

1.2. Functors.

Definition 1.4. Let $\mathscr{A}, \mathscr{B}$ be a categories. A *functor* $F : \mathscr{A} \to \mathscr{B}$ is the datum of

- a map $F : \mathrm{Obj}(\mathscr{A}) \to \mathrm{Obj}(\mathscr{B})$;
- for any $X, Y \in \mathscr{A}$, a map $F : \mathrm{Hom}_{\mathscr{A}}(X, Y) \to \mathrm{Hom}_{\mathscr{B}}(F(X), F(Y))$

which satisfy the following conditions:

- for any $X \in \mathrm{Obj}(\mathscr{A})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$;

- for any $X, Y, Z \in \mathrm{Obj}(\mathscr{A})$ and any morphisms $f : X \to Y$, $g : Y \to Z$, $F(g \circ f) = F(g) \circ F(f)$.

1.3. **Morphisms of functors.**

Definition 1.5. Let $\mathscr{A}, \mathscr{B}$ be categories.

- Let $F, G : \mathscr{A} \to \mathscr{B}$ be functors. A *morphism of functors* $\varphi : F \to G$ is the datum, for any $X \in \mathrm{Obj}(\mathscr{A})$, of a morphism $\varphi_X : F(X) \to G(X)$ such that for any morphism $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\varphi_X}\downarrow & & \downarrow{\scriptstyle\varphi_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

- Let $F, G : \mathscr{A} \to \mathscr{B}$ be functors. A morphism of functors $\varphi : F \to G$ is called an isomorphism if there exists a morphism of functors $\psi : G \to F$ such that $\psi \circ \varphi = \mathrm{id}_F$ and $\varphi \circ \psi = \mathrm{id}_G$, or in other words if φ_X is an isomorphism for any $X \in \mathrm{Obj}(\mathscr{A})$.
- A functor $F : \mathscr{A} \to \mathscr{B}$ is called an *equivalence of categories* if there exists a functor $G : \mathscr{B} \to \mathscr{A}$ and isomorphisms of functors $F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathscr{B}}$ and $G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathscr{A}}$.

Theorem 1.6. *Let $\mathscr{A}, \mathscr{B}$ be categories, and let $F : \mathscr{A} \to \mathscr{B}$ be a functor. Then F is an equivalence of categories iff it is:*

- *fully faithful, i.e. for any $X, Y \in \mathrm{Obj}(\mathscr{A})$ the map*

$$F : \mathrm{Hom}_{\mathscr{A}}(X, Y) \to \mathrm{Hom}_{\mathscr{B}}(F(X), F(Y))$$

  is a bijection;
- *essentially surjective, i.e. for any $Y \in \mathrm{Obj}(\mathscr{B})$ there exist $X \in \mathrm{Obj}(\mathscr{A})$ and an isomorphism $F(X) \cong Y$.*

1.4. **Adjoint functors.**

Definition 1.7. Let $\mathscr{A}, \mathscr{B}$ be categories, and let $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$ be functors. An *adjunction* between F and G is the datum of morphisms of functors

$$\varphi : \mathrm{id}_{\mathscr{A}} \to G \circ F, \quad \psi : F \circ G \to \mathrm{id}_{\mathscr{B}}$$

such that for any $X, Y \in \mathrm{Obj}(\mathscr{A})$ the compositions

$$\mathrm{Hom}_{\mathscr{B}}(F(X), Y) \xrightarrow{G} \mathrm{Hom}_{\mathscr{A}}(GF(X), G(Y)) \xrightarrow{(-)\circ\varphi_X} \mathrm{Hom}_{\mathscr{A}}(X, G(Y))$$

and

$$\mathrm{Hom}_{\mathscr{A}}(X, G(Y)) \xrightarrow{F} \mathrm{Hom}_{\mathscr{B}}(F(X), FG(Y)) \xrightarrow{\varphi_Y\circ(-)} \mathrm{Hom}_{\mathscr{B}}(F(X), Y)$$

are mutually inverse bijections.

2. Additive and abelian categories

2.1. **Additive categories and functors.**

Definition 2.1. A category $\mathscr{A}$ is called:

- *pre-additive* if, for any $X, Y \in \mathrm{Obj}(\mathscr{A})$, we are given on the set $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ a structure a abelian group such that the compositions laws $\circ$ are bilinear;

- *additive* if it is pre-additive and, in addition:
  - there exists a zero object $0 \in \mathrm{Obj}(\mathscr{A})$, i.e. an object such that
$$\mathrm{Hom}_{\mathscr{A}}(X,0) = 0 = \mathrm{Hom}_{\mathscr{A}}(0,X)$$
    for any $X \in \mathrm{Obj}(\mathscr{A})$;
  - for any $X, Y \in \mathscr{A}$ there exists an object $Z \in \mathrm{Obj}(\mathscr{A})$ and morphisms $i_X : X \to Z$, $i_Y : Y \to Z$, $p_X : Z \to X$, $p_Y : Z \to Y$ such that
$$p_X \circ i_X = \mathrm{id}_X, \quad p_Y \circ i_Y = \mathrm{id}_Y, \quad p_Y \circ i_X = 0, \quad p_X \circ i_Y = 0,$$
$$i_X \circ p_X + i_Y \circ p_Y = \mathrm{id}_Z.$$

Given $X, Y \in \mathrm{Obj}(\mathscr{A})$, an object Z as in the definition above is necessarily unique, and denoted $X \oplus Y$. If satisfies
$$\mathrm{Hom}_{\mathscr{A}}(W, X \oplus Y) = \mathrm{Hom}_{\mathscr{A}}(W,X) \oplus \mathrm{Hom}_{\mathscr{A}}(W,Y),$$
$$\mathrm{Hom}_{\mathscr{A}}(X \oplus Y, W) = \mathrm{Hom}_{\mathscr{A}}(X,W) \oplus \mathrm{Hom}_{\mathscr{A}}(Y,W)$$
for any $W \in \mathrm{Obj}(\mathscr{A})$.

Definition 2.2. Given pre-additive categories $\mathscr{A}, \mathscr{B}$, a functor $F : \mathscr{A} \to \mathscr{B}$ is called *additive* if for any $X, Y \in \mathrm{Obj}(\mathscr{A})$ the map
$$F : \mathrm{Hom}_{\mathscr{A}}(X,Y) \to \mathrm{Hom}_{\mathscr{B}}(F(X), F(Y))$$
is a group morphism.

If $\mathscr{A}, \mathscr{B}$ are additive categories and $F : \mathscr{A} \to \mathscr{B}$ is an additive functor, then we have
$$F(0) = 0$$
and
$$F(X \oplus Y) \cong F(X) \oplus F(Y)$$
for any $X, Y \in \mathscr{A}$.

2.2. **Abelian categories.**

Definition 2.3. Let $\mathscr{A}$ be an additive category, let $X, Y \in \mathrm{Obj}(\mathscr{A})$, and let $f \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$.

- A *kernel* of f is the datum an object $\ker(f) \in \mathrm{Obj}(\mathscr{A})$ and a morphism $i : \ker(f) \to X$ such that for any $Z \in \mathrm{Obj}(\mathscr{A})$ the following is an exact sequence of abelian groups:
$$0 \to \mathrm{Hom}_{\mathscr{A}}(Z, \ker(f)) \xrightarrow{i \circ (-)} \mathrm{Hom}_{\mathscr{A}}(Z,X) \xrightarrow{f \circ (-)} \mathrm{Hom}_{\mathscr{A}}(Z,Y).$$
- A *cokernel* of f is the datum an object $\mathrm{coker}(f) \in \mathrm{Obj}(\mathscr{A})$ and a morphism $p : Y \to \mathrm{coker}(f)$ such that for any $Z \in \mathrm{Obj}(\mathscr{A})$ the following is an exact sequence of abelian groups:
$$0 \to \mathrm{Hom}_{\mathscr{A}}(\mathrm{coker}(f), Z) \xrightarrow{(-) \circ p} \mathrm{Hom}_{\mathscr{A}}(Y,Z) \xrightarrow{(-) \circ f} \mathrm{Hom}_{\mathscr{A}}(X,Z).$$

If $f : X \to Y$ admits a cokernel, and if the morphism $Y \to \mathrm{coker}(f)$ admits a kernel, then this kernel is called the *image* of f, and is denoted $\mathrm{im}(f)$.

Definition 2.4. An *abelian* category is an additive category $\mathscr{A}$ in which each morphism admits a kernel and a cokernel, and such that for any $X, Y \in \mathrm{Obj}(\mathscr{A})$ and any $f \in \mathrm{Hom}_{\mathscr{A}}(X,Y)$ the natural map from the cokernel of the map $\ker(f) \to X$ to the kernel of the map $Y \to \mathrm{coker}(f)$ is an isomorphism.

If $\mathscr{A}$ is an abelian category, a morphism $f : X \to Y$ is said to be *injective* if $\ker(f) = 0$, and *surjective* if $\mathrm{coker}(f) = 0$.

Definition 2.5. Let $\mathscr{A}$ be an abelian category.

(1) An object $X \in \mathrm{Obj}(\mathscr{A})$ is said to be *simple* if it is nonzero and moreover there exists no nonzero injective morphism $f : Y \to X$ which is not an isomorphism.
(2) An object $X \in \mathrm{Obj}(\mathscr{A})$ is said to be *semisimple* if it is isomorphic to a (finite) direct sum of simple objects.
(3) The category $\mathscr{A}$ is said to be semisimple if any object of $\mathscr{A}$ is semisimple.

3. Exact functors

3.1. Definition.

Definition 3.1. Let $\mathscr{A}$ be an abelian category. A sequence of objects and morphisms in $\mathscr{A}$:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1}$$

is called an exact sequence if for any $i \in \{2, \cdots, n\}$ we have $f_i \circ f_{i-1} = 0$, and moreover the natural map $\mathrm{im}(f_{i-1}) \to \ker(f_i)$ is an isomorphism.

Definition 3.2. Let $\mathscr{A}, \mathscr{B}$ be abelian categories, and let $F : \mathscr{A} \to \mathscr{B}$ be a functor. Then F is called

- *left exact* if for any exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathscr{A}$, the collection

$$0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is an exact sequence in $\mathscr{B}$;
- *right exact* if for any exact sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathscr{A}$, the collection

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0$$

is an exact sequence in $\mathscr{B}$;
- *exact* if it is both left exact and right exact, or equivalently if for any exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathscr{A}$, the collection

$$0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0$$

is an exact sequence in $\mathscr{B}$.

3.2. **Projective and injective objects.** Let $\mathscr{A}$ be an abelian category.

Definition 3.3. An object X of $\mathscr{A}$ is said to be:

- *projective* if the functor
$$\mathrm{Hom}_{\mathscr{A}}(X,-):\mathscr{A}\to\mathsf{Mod}_{\mathbb{Z}}$$
is exact;
- *injective* if the functor
$$\mathrm{Hom}_{\mathscr{A}}(-,X):\mathscr{A}\to\big(\mathsf{Mod}_{\mathbb{Z}}\big)^{\mathrm{op}}$$
is exact.

Definition 3.4.

- One says that the category $\mathscr{A}$ *admits enough projectives* if for any $X\in\mathrm{Obj}(\mathscr{A})$ there exists an object Y in $\mathscr{A}$ which is projective and a surjective morphism $Y\to X$.
- One says that the category $\mathscr{A}$ *admits enough injectives* if for any $X\in\mathrm{Obj}(\mathscr{A})$ there exists an object Y in $\mathscr{A}$ which is injective and an injective morphism $X\to Y$.

4. Grothendieck group

4.1. **For additive categories.** If $\mathscr{A}$ is an additive category, the *split Grothendieck group* $K^0_{\oplus}(\mathscr{A})$ is the quotient of the free abelian group generated by symbols $[A]$ where A runs over isomorphism classes of objects of $\mathscr{A}$, by the relations
$$[B]=[A]+[C]$$
if $B\cong A\oplus C$.

This abelian group has the property that if M is an abelian group and f is a function from the set of objects of $\mathscr{A}$ to M which has the following properties:

- $f(A)=f(A')$ if $A\cong A'$;
- $f(A\oplus A')=f(A)+f(A')$,

then f factors through a group morphism $K^0_{\oplus}(\mathscr{A})\to M$.

4.2. **For abelian categories.** If $\mathscr{A}$ is an abelian category, the *Grothendieck group* $K^0(\mathscr{A})$ is the quotient of the free abelian group generated by symbols $[A]$ where A runs over isomorphism classes of objects of $\mathscr{A}$, by the relations
$$[B]=[A]+[C]$$
if there exists an exact sequence
$$0\to A\to B\to C\to 0.$$

This abelian group has the property that if M is an abelian group and f is a function from the set of objects of $\mathscr{A}$ to M which has the following properties:

- $f(A)=f(A')$ if $A\cong A'$;
- for any exact sequence $0\to A\to B\to C\to 0$, we have $f(B)=f(A)+f(C)$,

then f factors through a group morphism $K^0(\mathscr{A})\to M$.

A favorable case is when every object of $\mathscr{A}$ has finite length. In this case, if $(L_i:i\in I)$ is a system of representatives of isomorphism classes of simple objects, then $([L_i]:i\in I)$ is a basis of $K^0(\mathscr{A})$.

4.3. **For triangulated categories.** (This subsection is meant to be read after Lecture 5.)

If $\mathscr{A}$ is a triangulated category, the *Grothendieck group* $K^0_\Delta(\mathscr{A})$ is the quotient of the free abelian group generated by symbols $[A]$ where A runs over isomorphism classes of objects of $\mathscr{A}$, by the relations

$$[B] = [A] + [C]$$

if there exists a distinguished triangle

$$A \to B \to C \xrightarrow{+1} .$$

This abelian group has the property that if M is an abelian group and f is a function from the set of objects of $\mathscr{A}$ to M which has the following properties:

- $f(A) = f(A')$ if $A \cong A'$;
- for any distinguished triangle $A \to B \to C \xrightarrow{+1}$, we have $f(B) = f(A) + f(C)$,

then f factors through a group morphism $K^0_\Delta(\mathscr{A}) \to M$.

If $\mathscr{A}$ is a triangulated category equipped with a bounded t-structure with heart $\mathscr{A}^\heartsuit$, then it is know that the embedding $\mathscr{A}^\heartsuit \subset \mathscr{A}$ induces a canonical isomorphism

$$K^0(\mathscr{A}^\heartsuit) \xrightarrow{\sim} K^0_\Delta(\mathscr{A}).$$