Lecture 11: main exercises

Exercise 11.1. Recall the sheaves

\[ \mathcal{P}_{k,\ell} := \mathcal{O}_{Gr_{GL_n}} \otimes \det(L_0/L)^{\ell} \bigl[ \frac{1}{2} k(n-k) \bigr] \]

as well as the convolution varieties

\[ Gr^{(k_1, k_2)}_{GL_n} := \{ L_2 \subset L_1 \subset L_0 : tL_0 \subset L_1, tL_1 \subset L_2 \} \].

(a) Show that \( \mathcal{P}_{1,\ell_1} \ast \mathcal{P}_{1,\ell_2} \cong \mathcal{P}_{1,\ell_2} \ast \mathcal{P}_{1,\ell_1} \) whenever \( |\ell_1 - \ell_2| \leq 1 \).

(b) Consider the following space

\[ W_k := \{ L_2 \subset L_1 \subset L_1' \subset L_0 : tL_0 \subset L_1, tL_1' \subset L_2 \} \]

where \( k \leq n-2 \). Denote by \( D \subset W_k \) the locus where \( \{ tL_0 \subset L_2 \} \). Describe the ideal sheaf \( \mathcal{I}_D \) in terms of line bundles.

(c) Show that \( \mathcal{P}_{1,\ell_1} \ast \mathcal{P}_{k,\ell_2} \cong \mathcal{P}_{k,\ell_2} \ast \mathcal{P}_{1,\ell_1} \) whenever \( |\ell_1 - \ell_2| \leq 1 \).

It may help to use the following commutative diagram

[Diagram]

where the maps \( \pi_1 \) and \( \pi_2 \) are given by forgetting \( L_1' \) and \( L_1 \) respectively.
Lecture 11: additional exercises

Exercise 11.2. (a) Consider the following space

\[ W^m_k := \{ L_2 \subseteq L_1 \subseteq L'_1 \subseteq L_0 : tL_0 \subseteq L_1, tL'_1 \subseteq L_2 \} \]

where \( k \leq n - 2m \). Denote by \( D \subseteq W^m_k \) the locus where \( t : L_0/L'_1 \to L_1/L_2 \) is not an isomorphism. Describe the ideal sheaf \( \mathcal{I}_D \) in terms of line bundles.

(b) Show that \( \mathcal{P}_{m,t_1} \ast \mathcal{P}_{k,t_2} \cong \mathcal{P}_{k,t_2} \ast \mathcal{P}_{m,t_1} \) whenever \( |t_1 - t_2| \leq 1 \).

Notice that you can assume \( m \leq k \). It may then help to use the following commutative diagram

\[
\begin{array}{ccc}
W^m_{k-m} & \xrightarrow{\pi_2} & \text{Gr}_{GL_n}^{(k,m)} \\
\downarrow{\pi_1} & & \downarrow{m_2} \\
\text{Gr}_{GL_n}^{(m,k)} \downarrow{m_1} & & \text{Gr}_{GL_n} \downarrow{\omega'_m + \omega'_k} \\
& & \\
\end{array}
\]

where the maps \( \pi_1 \) and \( \pi_2 \) are given by forgetting \( L'_1 \) and \( L_1 \) respectively.