Lecture 12: main exercises

Exercise 12.1. Recall the sheaves

$$\mathcal{P}_{k,\ell} := \mathcal{O}_{\mathsf{Gr}^k_{GL_n}} \otimes \det(L_0/L)^\ell[\frac{1}{2}k(n-k)]$$

as well as the convolution varieties

$$\mathsf{Gr}^{(k_1,k_2)}_{GL_n} := \{L_2 \overset{k_2}{\subset} L_1 \overset{k_1}{\subset} L_0 : tL_0 \subset L_1, tL_1 \subset L_2\}.$$

Show that for $k \in [0,n]$ and $\ell \in \mathbb{Z}$ we have exact triangles

$$\mathcal{P}_{k-1,\ell} * \mathcal{P}_{k+1,\ell} \to \mathcal{P}_{k,\ell+1} * \mathcal{P}_{k,\ell-1} \to \mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell}$$
$$\mathcal{P}_{k,\ell} * \mathcal{P}_{k,\ell} \to \mathcal{P}_{k,\ell-1} * \mathcal{P}_{k,\ell+1} \to \mathcal{P}_{k+1,\ell} * \mathcal{P}_{k-1,\ell}$$

in the derived category of Gr_{GL_n} (here for notational convenience $\mathcal{P}_{-1,\ell}$ and $\mathcal{P}_{n+1,\ell}$ are interpreted as zero).

You might want to first assume $\ell = 0$ to (at least) simplify notation. The general case should follow fairly easily.