Lecture 1: main exercises

Exercise 1.1. Take $G = GL_n$ and consider the convolution map

$$m : \mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_1^\vee} \to \mathsf{Gr}_{\leq 2\omega_1^\vee}.$$

Recall the lattice description

$$\mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_1^\vee} = \{L_2 \overset{1}{\subset} L_1 \overset{1}{\subset} L_0 : tL_0 \subset L_1, tL_1 \subset L_2\}$$

and use it to help you with the following questions.

(a) Describe the two $G_{\mathcal{O}}$-orbits of $\mathsf{Gr}_{\leq 2\omega_1^\vee}$. What is the codimension of the smaller orbit?

(b) Identify the different fibers of the map $m : \mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_1^\vee} \to \mathsf{Gr}_{\leq 2\omega_1^\vee}$.

(c) Check that this map is semi-small.

Exercise 1.2. Do the same for the map

$$m : \mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_{n-1}^\vee} \to \mathsf{Gr}_{\leq \omega_1^\vee + \omega_{n-1}^\vee}.$$

Recall that the lattice description of $\mathsf{Gr}_{\omega_{n-1}^\vee}$ is

$$\mathsf{Gr}_{\omega_{n-1}^\vee} = \{tL_0 \overset{1}{\subset} L \overset{n-1}{\subset} L_0\}.$$

Lecture 1: additional exercises

Exercise 1.3. This exercise gives more practice computing with fibers and orbits, in the setting of finite Grassmannians (this isn't related to affine Grassmannians). Let $G = GL_4$, and let Q be the subgroup of all invertible matrices of the form

$$Q = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}.$$

Then Q is the stabilizer in G of a the plane P spanned by the first basis vectors e_1 and e_2.

For any $0 \leq k \leq 4$, Q acts on the Grassmannian $\mathsf{Gr}(k,4)$ of k-planes (i.e. k-dimensional subspaces) in $\mathbb{C}^4$. For each $d \geq 0$, let

$$\mathsf{Gr}_d(k,4) = \{M \subset \mathbb{C}^4 \mid \dim(M) = k,\ \dim(M \cap P) = d\}.$$

(a) Prove that $\mathsf{Gr}_d(k,4)$ is preserved by the Q action, for each k and d. Indeed, these are actually the Q orbits on $\mathsf{Gr}(k,4)$.

(b) For each $0 \leq k \leq 4$, how does $\mathsf{Gr}(k,4)$ split into Q orbits? That is, which orbits are non-empty? Which orbits are contained in the closure of the others?

(c) Let Y be the following space:

$$Y = \{(L,M) \in \mathsf{Gr}(1,4) \times \mathsf{Gr}(2,4) \mid L \subset (M \cap P)\}.$$

What is the image of Y under the forgetful map $\pi : Y \to \mathsf{Gr}(2,4)$, and how does it decompose into Q orbits? Identify the different fibers of π. Is π semismall?

(d) What is the image of Y under the forgetful map $\rho : Y \to \mathsf{Gr}(1,4)$? What is the fiber over each point in the image? Deduce that Y is smooth, and compute its dimension.

Aside with spoiler: Indeed, Y is a resolution of singularities for the closure of the orbit $\mathsf{Gr}_1(2,4)$.