Lecture 2: main exercises

Exercise 2.1. Consider the embeddings

$$i : \{0\} \to \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$$

and

$$j : U := \mathbb{A}^2 \setminus \{0\} \to \mathbb{A}^2.$$

Note that U is not affine. Identify in an explicit way the following sheaves. For example, if it is a sheaf on $\mathbb{A}^2$, identify it as an explicit $\mathbb{C}[x, y]$-module.

(a) $i^*(\mathcal{I}_0)$ where $\mathcal{I}_0$ is the ideal sheaf (x, y) of $\{0\} \subset \mathbb{A}^2$.

(b) $i_*(\mathcal{O}_0)$. This is the structure sheaf of $\{0\} \subset \mathbb{A}^2$, and we also abusively denote it $\mathcal{O}_0$.

(c) $i^*(\mathcal{O}_0)$.

(d) $i_*(\mathcal{O}_0)^\vee$.

(e) $j^*(\mathcal{O}_0)$.

(f) $j_*(\mathcal{O}_U)$.

(g) $j_*(j^*(\mathcal{O}_0) \otimes \mathcal{O}_U)$ and $\mathcal{O}_0 \otimes j_*(\mathcal{O}_U)$.

Exercise 2.2. Take $G = GL_n$ and consider the convolution map

$$m : \mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_1^\vee} \to \mathsf{Gr}_{\leq 2\omega_1^\vee}.$$

Consider the small orbit $\mathsf{Gr}_{\omega_2^\vee} \subset \mathsf{Gr}_{\leq 2\omega_1^\vee}$ and denote $D = m^{-1}(\mathsf{Gr}_{\omega_2^\vee})$.

(a) Find an appropriate bundle V with a section s which vanishes precisely along D.

(b) Use this to compute (describe) the conormal bundle of $D \subset \mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_1^\vee}$.

(c) Compute the relative cotangent bundle of $D \to \mathsf{Gr}_{\omega_2^\vee}$ and compare with the answer above.

Lecture 2: additional exercises

Exercise 2.3. Repeat the previous exercise with the map

$$m : \mathsf{Gr}_{\omega_1^\vee} \widetilde{\times} \mathsf{Gr}_{\omega_{n-1}^\vee} \to \mathsf{Gr}_{\leq \omega_1^\vee + \omega_{n-1}^\vee}.$$

Exercise 2.4. Let $\pi : V \to X$ be a rank r vector bundle over a scheme X. Find a rank r vector bundle W on V and a section of W which vanishes exactly along the zero section $X \subset V$. What is the normal bundle of $X \subset V$?