Lecture 3: main exercises

Exercise 3.1. Let $R = \mathbb{C}[x_1, x_2]$, where x_1 and x_2 have weight 1. Let S be the R-module defined by $S = R/(x_1, x_2)$.

Let $\Lambda = \mathbb{C}[\varepsilon_1, \varepsilon_2]$, where ε_1 and ε_2 are *graded commutative* variables of weight 1 and degree -1. That is, they obey

$$\varepsilon_1^2 = \varepsilon_2^2 = 0, \qquad \varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1.$$

Let $K^\bullet := \Lambda \otimes R$. We write $\varepsilon_i x_j$ for $\varepsilon_i \otimes x_j$, etc. We equip $K^\bullet$ with an R-linear differential d determined by

$$d(1) = 0, \qquad d(\varepsilon_i) = x_i, \qquad d(\varepsilon_i\varepsilon_j) = \varepsilon_i x_j - \varepsilon_j x_i.$$

(a) Show that S has a projective resolution $K^\bullet \to S$.

(b) For each of the following graded R-modules M, find a quasi-isomorphism from $\mathrm{Hom}_R^\bullet(K^\bullet, M) \in \mathrm{Ch}_{\mathbb{C}}^{\mathrm{gr}}$ to a complex with zero differential.

Use this to compute the cohomology of $\mathrm{RHom}_R(S, M) \in \mathrm{DMod}_{\mathbb{C}}^{\mathrm{gr}}$, or in other words, to compute the groups $\mathrm{Ext}^k(S, M)$).

(i) $M = R$,

(ii) $M = S$,

(iii) $M = R/(x_1)$.

Exercise 3.2. Let $\Lambda^\vee = \mathbb{C}[\theta_1, \theta_2]$, where θ_1 and θ_2 are graded commutative variables of weight -1 and degree 1.

Find an R-linear action of $\Lambda^\vee$ on $K^\bullet$ which is compatible with the gradings (i.e. the action of θ_i should decrease weight by 1 and increase degree by 1). Show that the induced algebra morphism $\Lambda^\vee \to \mathrm{Hom}_R^\bullet(K^\bullet, K^\bullet)$ is also a quasi-isomorphism of complexes.

(Warmup: first do this for the 1-variable case.)

Lecture 3: additional exercises

Exercise 3.3. Let $R = \mathrm{Sym}^\bullet(V)$, where V is a finite-dimensional vector space with weight 1. Also let $\Lambda = \mathrm{Sym}^\bullet(V[1])$, interpreted in the graded commutative sense. That is, Λ is isomorphic as an ungraded algebra to the exterior algebra $\Lambda^\bullet(V)$.

(a) Find a differential on $K^\bullet := \Lambda \otimes R$ such that $S = R/(V)$ has a projective resolution $K^\bullet \to S$.

(b) For each of the following graded R-modules M, find a quasi-isomorphism from $\mathrm{Hom}^\bullet_R(K^\bullet, M) \in \mathrm{Ch}^{\mathrm{gr}}_{\mathbb{C}}$ to a complex with zero differential.

Use this to compute the cohomology of $\mathrm{RHom}_R(S, M) \in \mathrm{DMod}^{\mathrm{gr}}_{\mathbb{C}}$, or in other words, to compute the groups $\mathrm{Ext}^k(S, M)$.

(i) $M = R$,

(ii) $M = S$,

(iii) $M = R/(W)$, where $W \subset V$ is a subspace.

(c) Let $\Lambda^\vee = \mathrm{Sym}^\bullet(V^*[-1])$, interpreted in the graded commutative sense, where V^* is the linear dual of V. Find an R-linear action of $\Lambda^\vee$ on $K^\bullet$ which is compatible with the gradings. Show that the induced algebra morphism $\Lambda^\vee \to \mathrm{Hom}^\bullet_R(K^\bullet, K^\bullet)$ is also a quasi-isomorphism of complexes.

Exercise 3.4. More practice computing Exts. Let $R = \mathbb{C}[x_1, x_2]$. Let $M = R/(x^2, xy, y^2)$.

(a) Find a free resolution of M.

(b) Compute $\mathrm{Ext}^k(M, S)$ for all $k \geq 0$.

(c) Compute $\mathrm{Ext}^k(M, M)$ for all $k \geq 0$.

Exercise 3.5. More practice computing Exts. Let $A = \mathbb{C}[y]/(y^4)$. For each $0 \leq i \leq 4$, let $M_i = A/(y^i)$, an i-dimensional indecomposable A-module.

(a) For each i, find a free resolution of M_i.

(b) Compute $\mathrm{Ext}^k(M_1, M_2)$ and $\mathrm{Ext}^k(M_3, M_2)$ for all $k \geq 0$.

(c) Compute $\mathrm{Ext}^k(M_2, M_1)$ and $\mathrm{Ext}^k(M_2, M_3)$ for all $k \geq 0$.

(d) Compute $\mathrm{Ext}^k(M_3, M_3)$ for all $k \geq 0$.