Lecture 5: main exercises

Exercise 5.1. Recall the blowup $\pi : Y = \mathrm{Bl}_0 \mathbb{C}^2 \to \mathbb{C}^2$ as well as the description

$$\mathrm{Bl}_0 \mathbb{C}^2 \cong \{(0 \overset{1}{\subset} V \overset{1}{\subset} \mathbb{C}^2, v \in V)\}$$

where π is the map which forgets V. Additionally there is the map $p : Y \to \mathbb{P}^1$ which forgets v, and the inclusion $\mathbb{P}^1 \to Y$ as the locus where $v = 0$ (this is the exceptional divisor).

(a) In the lecture we computed $R\pi_* \mathcal{O}_Y$ by factoring π as the composition

$$Y \xrightarrow{i} \mathbb{P}^1 \times \mathbb{C}^2 \xrightarrow{\widetilde{\pi}} \mathbb{C}^2.$$

Use that method to calculate $R\pi_* p^* \mathcal{O}_{\mathbb{P}^1}(\ell)$ where $\ell = -1$. What about when $\ell = 1$?

(b) Find a line bundle on Y with a section which vanishes exactly along $\mathbb{P}^1 \subset Y$.

(c) Use this to identify the normal bundle $N_{\mathbb{P}^1/Y}$ and ideal sheaf $\mathscr{I}_{\mathbb{P}^1/Y}$ of $\mathbb{P}^1 \subset Y$.

(d) Use the short exact sequence $0 \to \mathscr{I}_{\mathbb{P}^1/Y} \to \mathcal{O}_Y \to \mathcal{O}_{\mathbb{P}^1} \to 0$ to check your answers for $R\pi_* p^* \mathcal{O}_{\mathbb{P}^1}(\ell)$ above.

(e) Extra challenge: try to do the same with $\pi : \mathrm{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$ for $n > 2$?

(f) Use the embedding i (or any other method) to identify explicitly the dualizing sheaf ω_Y of Y.

(g) Compute $\mathbb{D}(\mathcal{O}_Y)$ and use the calculations above to confirm that $R\pi_* \mathbb{D}(\mathcal{O}_Y) \cong \mathbb{D}(R\pi_* \mathcal{O}_Y)$.

Lecture 5: additional exercises

Exercise 5.2. Let $\pi : V \to X$ be a rank r vector bundle over a scheme X and let $\sigma : X \to V$ be the zero section.

(a) Compute $L\sigma^*\sigma_*(\mathcal{O}_X) \in D_{coh}(X)$

(b) Compute $\mathscr{H}om(\sigma_*\mathcal{O}_X, \sigma_*\mathcal{O}_X) \in D_{coh}(V)$.