Derived Categories

- Flag $R \rightsquigarrow \mathrm{Mod}_R = \{R\text{-modules}\}$
  $\qquad\qquad \cap$
  $\quad \mathrm{DMod}_R = \{\text{"derived" } R\text{-modules}\}$

Analogy: $\mathrm{Mod}_R : \mathrm{DMod}_R :: \mathbb{R} : \mathbb{C}$

simpler to define — simpler in terms of formal properties

- $\mathbb{C}$ solves the problem of polynomials not having solutions, DMod_R solves the problem of functors not being exact.
- $L \xrightarrow{\phi} M \xrightarrow{\psi} N$ is a short exact sequence in Mod_R if

$$(*)\quad \begin{tikzcd} & M \arrow[dr, "\psi"] \arrow[drr, bend left] & & \\ L \arrow[ur, "\phi"] \arrow[dr] & & N \arrow[r, dashed, "\exists!"] & Q \\ & 0 \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}\quad \text{pushout}$$

is a pushout and a pullback
(i.e. $\psi = \mathrm{cok}(\phi)$ and $\phi = \ker(\psi)$)
- $F : \mathrm{Mod}_R \to \mathrm{Mod}_A$ is right/left exact if

$$\begin{tikzcd} & F(M) \arrow[dr, "F(\psi)"] & \\ F(L) \arrow[ur, "F(\phi)"] \arrow[dr] & & F(N) \\ & 0 \arrow[ur] & \end{tikzcd}$$

is a pushout/pullback whenever $(*)$ is.
- exact := left and right exact

Two descriptions of DMod_R:
(1) Explicit construction (existence)
(2) Universal property (uniqueness)

Explicit Construction

- $\mathrm{Ch}_R = \{\text{cochain complexes}\}$
  $\left(\begin{array}{l}\text{graded } R\text{-module } M^\cdot \text{ w/ } d : M^\cdot \circlearrowleft \text{ s.t.} \\ d(M^k) \subset M^{k+1} \text{ and } d^2 = 0\end{array}\right)$
- cohomology of $M^\cdot$ is $H^\cdot M^\cdot = \ker d / \mathrm{im}\, d$
  (graded via $H^k M^\cdot = \ker d^k / \mathrm{im}\, d^{k+1}$)
- A cochain map $f : M^\cdot \to N^\cdot$ is a graded R-module homomorphism s.t. $df = fd$.
- f then induces $H^\cdot f : H^\cdot M^\cdot \to H^\cdot N^\cdot$, and is a quasi-isomorphism if $H^\cdot f$ is an isomorphism

Def Letting $Q = \{\text{quasi-isomorphisms}\}$, we set $\mathrm{DMod}_R = \mathrm{Ch}_R[Q^{-1}]$ (i.e. we freely adjoin inverses to quasi-isomorphisms)
- Not obvious how to work with this."
- But clearly $M^\cdot \mapsto H^\cdot M^\cdot$ descends to a functor $\mathrm{DMod}_R \to \mathrm{Mod}_R^{gr}$, and one can show

$$\mathrm{Mod}_R \xhookrightarrow[\deg 0]{} \mathrm{Ch}_R \longrightarrow \mathrm{DMod}_R$$

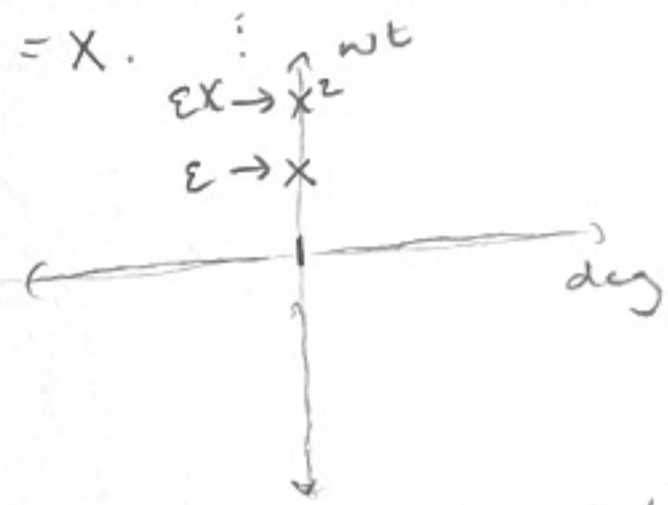
is fully faithful w/ image $\{X \mid H^k(X) = 0 \text{ for } k \neq 0\}$.
- Key fact: a left/right exact functor like
  $$\mathrm{Hom}_R(M,-) : \mathrm{Mod}_R \to \mathrm{Mod}_{\mathbb{Z}}$$
  "induces" an exact "derived functor"
  $$R\mathrm{Hom}_R(M,-) : \mathrm{DMod}_R \to \mathrm{DMod}_{\mathbb{Z}}$$

Variants: $D^b / D^- / D^+ \mathrm{Mod}_R \subset \mathrm{DMod}_R$
$= \{X \mid H^k(X) = 0 \text{ for } |k| / k / -k \gg 0\}$

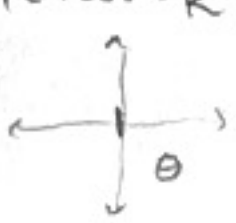
Computing derived functors

- $P \in \mathrm{Mod}_R$ projective if $\mathrm{Hom}_R(P,-)$ is exact (e.g. if P is free)
- A projective resolution of M is a q-iso $P^\cdot \to M$ w/ P^k projective $\forall k$ and zero $\forall k > 0$.
- Then $R\mathrm{Hom}_R(M, X)$ is represented by $\mathrm{Hom}_R^\cdot(P^\cdot, N^\cdot) \in \mathrm{Ch}_{\mathbb{Z}}$ for any representative $N^\cdot$ of X.
- Here $\deg \phi = k$ if $\phi(P^\ell) \subset N^{\ell+k}$ $\forall \ell$, and $d(\phi) = d \circ \phi - (-1)^k \phi \circ d$ if $\deg \phi = k$.

Ex $R = \mathbb{C}[x]$ as a graded ring ($\mathrm{wt}\, x = 1$),
$\mathrm{Mod}_R^{gr} = \{\text{graded } R\text{-modules } M = \oplus M_k\}$
- $S = R/(x)$ has resolution $\Lambda \otimes R \to S$, where $\Lambda = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ w/ $\deg \varepsilon = -1$, $\mathrm{wt}\, \varepsilon = 1$, and $d(\varepsilon) = x$.



- Then $R\mathrm{Hom}_R(S,S) \simeq \mathrm{Hom}_R^\cdot(\Lambda \otimes R, S)$
  $\simeq \mathrm{Hom}_{\mathbb{C}}^\cdot(\Lambda, S)$
  $\simeq \Lambda^* \otimes S$



shift deg / wt ↓ ↓

- Likewise $R\mathrm{Hom}_R(S,R) \simeq \Lambda^* \otimes R \underset{?}{\simeq} \mathbb{C}[-1]\langle -1 \rangle$

$d(1) = -\theta x$