t-structures

Q: Given $X \in DMod_R^{gr}$, can we define $H^\bullet X \in Mod_R^{gr}$ intrinsically? (i.e. w/o choosing a complex $M^\bullet$ which represents X)

A: Yes, and this leads to a robust notion of "generalized cohomology"

Some terminology: SES's in $DMod_R$ are called exact triangles, cokernels are cones or cofibers, kernels are fibers.

Warning: these behave very differently in $DMod_R$ than Mod_R. E.g. let

$$X \xrightarrow{f} Y \xrightarrow{\psi} Z$$

be an exact triangle, and note that $M^\bullet \mapsto M[1]^\bullet$ (where $M[1]^k = M^{k+1}$) descends to $DMod_R$. Then $\exists$ morphism $Z \xrightarrow{\partial} X[1]$ s.t. $Y \xrightarrow{\psi} Z \xrightarrow{\partial} X[1]$ and $Z[-1] \xrightarrow{\partial[-1]} X \xrightarrow{f} Y$ are also exact triangles (i.e. we can "rotate triangles").

- Nonetheless, $Ch_R \to DMod_R$ takes SES's to exact triangles.

Def A t-structure on $\mathcal{C} = DMod_R$ is a full subcategory $\mathcal{C}^{\le 0}$ $(= \mathcal{C}_t^{\le 0})$ such that

(1) $\mathcal{C}^{\le 0} \hookrightarrow \mathcal{C}$ has a right adjoint $\tau^{\le 0} : \mathcal{C} \to \mathcal{C}^{\le 0}$, and

(2) $\mathcal{C}^{\le 0}$ is closed under extensions ($X, Z \in \mathcal{C}^{\le 0}$ and $X \to Y \to Z$ an exact triangle $\Rightarrow Y \in \mathcal{C}^{\le 0}$)

Ex The standard t-structure has

$$DMod_R^{\le 0} = \{X \mid H^k X \cong 0 \text{ for } k > 0\},$$

with $\tau^{\le 0}$ induced by

$$M^\bullet \mapsto \tau^{\le 0} M^\bullet := \dots \to M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} \ker d^0 \to 0 \to \dots$$

- One can show in general that

(3) the inclusion of $\mathcal{C}^{\ge 1} = \{Y \mid Hom_{\mathcal{C}}(X,Y) = 0 \ \forall X \in \mathcal{C}^{\le 0}\}$ has a left adjoint $\tau^{\ge 1} : \mathcal{C} \to \mathcal{C}^{\ge 1}$

(4) $\mathcal{C}^{\ge 1}$ is closed under extensions

(5) $\mathcal{C}^{\le 0}$ is closed under $[1]$ and $\mathcal{C}^{\ge 1}$ is closed under $[-1]$.

- Setting $\mathcal{C}^{\le n} = \mathcal{C}^{\le 0}[-n]$ and $\mathcal{C}^{\ge n} = \mathcal{C}^{\ge 1}[1-n]$, we have a filtration

$$\dots \subset \mathcal{C}^{\le -1} \subset \mathcal{C}^{\le 0} \subset \mathcal{C}^{\le 1} \subset \dots \subset \mathcal{C}$$

- The adjoints $\tau^{\le n}$ come w/ counits $\tau^{\le n} \to id_{\mathcal{C}}$, so every $X \in \mathcal{C}$ inherits a filtration

$$\dots \to \tau^{\le -1} X \to \tau^{\le 0} X \to \tau^{\le 1} X \to \dots \to X.$$

- Common way to study filtered objects: look at their associated graded objects (i.e. successive quotients)
- One can show that

$$Cone(\tau^{\le -1} X \to \tau^{\le 0} X) \cong \tau^{\ge 0} \tau^{\le 0}(X),$$

and that this belongs to the heart $\mathcal{C}^\heartsuit := \mathcal{C}^{\le 0} \cap \mathcal{C}^{\ge 0}$.

Ex For the std t-structure

$DMod_R^{\ge 0} = \{X \mid H^k X = 0 \text{ for } k < 0\}$ and

$DMod_R^\heartsuit = \{X \mid H^k X = 0 \text{ for } k \ne 0\} \cong Mod_R$,

and $\tau^{\ge 0}\tau^{\le 0}(X) \cong H^0 X$. (exercise)

- With this in mind we call

$$H_t^n := \tau^{\ge n}\tau^{\le n} : \mathcal{C} \to \mathcal{C}^\heartsuit$$

the nth "t-cohomology" functor.

Key facts: (1) $\mathcal{C}^\heartsuit$ is abelian (exactness behaves same as in Mod_R)

(2) if $X \to Y \to Z$ is an exact triangle then $H_t^n(X) \to H_t^n(Y) \to H_t^n(Z)$ is exact at $H_t^n(Y)$ (i.e. H_t^n is a "cohomological functor").

Summary: a subcategory $\mathcal{C}^{\le 0}$ w/ good closure properties defines a filtration, whose associated graded

$$H_t^\bullet : \mathcal{C} \to \mathcal{C}^{\heartsuit, gr}$$

generalizes the std cohomology.

Ex The Koszul t-structure

Exercises $R = Sym^\bullet(V)$, $\Lambda^\vee = Sym^\bullet(V^*[-1])$

$\rightsquigarrow \Lambda^\vee$ acts on $K^\bullet$, hence on $Hom_R^\bullet(K^\bullet, M^\bullet)$ for any $M^\bullet \in Ch_R^{gr}$.

- define $sh : Ch_{\mathbb{C}}^{gr} \circlearrowleft$ by $sh(M^\bullet) = M[n]^\bullet$ if $M^\bullet$ has weight $-n$.
- Setting $R^! := sh(\Lambda^\vee) \cong \Lambda^\bullet(V^*)$, it follows that $K := sh \circ Hom_R(K^\bullet, -)$ descends to a functor $DMod_R^{gr} \to DMod_{R^!}^{gr}$.

Thm (BGG) This restricts to an equivalence $DMod_R^{gr,fg} \simeq DMod_{R^!}^{gr,fg}$, the subcategories of X s.t. $H^\bullet X$ is finitely generated over R or $R^!$.

- The "Koszul" t-structure on $DMod_R^{gr,fg}$ is the image of the std t-structure on $DMod_{R^!}^{gr,fg}$ under this equivalence.