Perverse t-structures

- If a derived category $\mathcal{C}$ has some extra structure, it is natural to ask for a t-structure compatible with it

Ex For $D^b\mathrm{Coh}^{gr}_{\mathrm{Sym}^\bullet V}$, the Koszul heart is closed under (derived) tensor product, but the std heart is not.

- Given a variety X, the duality $\mathbb{D}$ of $D^b\mathrm{Coh}(X)$ does not preserve the std heart

Digression: Equivariant sheaves

- action $G \curvearrowright X$ of alg. group $\rightsquigarrow \mathrm{Coh}^G(X)$

(1) $X = \mathrm{Spec}\, R \rightsquigarrow G \curvearrowright R$ by alg. automorphisms and

$$\mathrm{Coh}^G(X) = \left\{ \begin{array}{l} M \text{ a } G\text{-rep and f.g. } R\text{-module} \\ \text{s.t. } R \otimes M \to M \text{ is } G\text{-linear} \end{array} \right\} = \mathrm{Mod}_R^{G,fg}$$

Ex $\mathrm{Rep}\, \mathbb{C}^\times = \{\text{graded vect. spaces}\}$

$$\mathbb{C}^\times \curvearrowright V \longmapsto V_k = \{ v \mid z \cdot v = z^k v \}$$

- R graded ring $\rightsquigarrow \mathrm{Coh}^{\mathbb{C}^\times}(X) \simeq \mathrm{Mod}_R^{gr,fg}$

(2) An equiv. vector bundle $E \to X$ is a ~~bundle~~ w/ an action $G \curvearrowright E$ which extends $G \curvearrowright X$ and is linear on f. bers. Sections $\mathcal{E}$ of E form a G-equiv. coherent sheaf.

(3) If G acts transitively then $X \cong G/G_p$, where G_p is the stabilizer of $p \in X$. Every $\mathcal{F} \in \mathrm{Coh}^G(X)$ is locally free, and

$$\begin{aligned} \mathrm{Coh}^G(X) &\longrightarrow \mathrm{Rep}\, G_p \\ E &\longmapsto E|_p \end{aligned}$$

is an equivalence.

- Now suppose X has finitely many G-orbits, all even-dimensional

Recall that $\mathrm{supp}\, \mathcal{F}$ = smallest closed $Z \subset X$ s.t. $\mathcal{F}|_{X \setminus Z} \simeq 0$.

Thm (Bez) $\mathcal{C} = D^b\mathrm{Coh}^G(X)$ has a "perverse" t-structure w/

$$\mathcal{C}_p^{\leq 0} = \{ \mathcal{F}^\bullet \mid \dim \mathrm{supp}\, H^k \mathcal{F}^\bullet \leq -k \;\; \forall k \} \; (\subset \mathcal{C}^{\leq 0})$$

($\dim \emptyset := -\infty$)

and $\mathcal{C}_p^{\geq 0} = \{ \mathcal{F}^\bullet \mid \mathbb{D}\mathcal{F}^\bullet \in \mathcal{C}_p^{\leq 0} \}$. Every $\mathcal{F}^\bullet \in \mathcal{P}^G_{coh}(X) := \mathcal{C}_p^{\heartsuit}$ has a composition series, i.e. a finite filtration s.t. each $\mathcal{F}_i^\bullet / \mathcal{F}_{i-1}^\bullet$ is simple. $\exists$ bijection

$$\begin{array}{c}\text{simples in} \\ \mathcal{P}^G_{coh}(X)\end{array} \longleftrightarrow \left\{ (\mathcal{O}, \mathcal{E}) \;\middle|\; \begin{array}{l} G\text{-orbit } \mathcal{O} \subset X, \\ \text{simple } \mathcal{E} \in \mathrm{Coh}^G(\mathcal{O}) \end{array} \right\},$$

with $\mathrm{supp}\, \mathcal{P}_{\mathcal{O},\mathcal{E}} = \overline{\mathcal{O}}$ and

$$\mathcal{P}_{\mathcal{O},\mathcal{E}}|_{\mathcal{O}} \simeq \mathcal{E}\left[\tfrac{\dim \mathcal{O}}{2}\right].$$

In particular, $\mathcal{P}^G_{coh}(X)$ is preserved by $\mathbb{D}$.

Ex -1: $X = pt \rightsquigarrow D^b\mathrm{Coh}^G(pt) \simeq D^b\mathrm{Rep}\, G$, perverse and std t-structures coincide

Ex 0: $\mathrm{Coh}^{GL_3}(\mathbb{P}^2) = \mathrm{Rep}\, P$, where $P = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$. Then $\mathcal{P}^{GL_3}_{coh}(\mathbb{P}^2) \simeq \mathrm{Coh}^{GL_3}(\mathbb{P}^2)$ and the former is $[1]$ of the latter in $D^b\mathrm{Coh}$.

Ex 1: $\mathrm{Coh}^{SL_2}(\mathbb{C}^2) = \left\{ \begin{array}{l} \text{f.g. } R = \mathbb{C}[x,y]\text{-modules w/} \\ \text{compatible } SL_2\text{-action} \end{array} \right\}$

- SL_2 orbits: $\{0\}$, $U = \mathbb{C}^2 \setminus \{0\}$
- $\omega_{\mathbb{C}^2} = R[2] \Rightarrow \mathbb{D} = R\mathcal{H}om(-, R)[2]$
- Lecture 1 exercises: $R\mathcal{H}om(S = i_{0*}\mathbb{C}, R) \simeq S[-2]$

hence $\mathbb{D}\, i_{0*}\mathbb{C} \simeq i_{0*}\mathbb{C}$, and $i_{0*}\mathbb{C}$ is simple perverse. (special case of $\mathbb{D}$ commuting w/ pushforward).

- $\mathbb{D}$ also commutes w/ forgetting equivariance, and $i_{0*}V_n$ is simple perverse for each simple SL_2-rep $V_n = \mathrm{Sym}^n \mathbb{C}^2$.
- Note that $R[1] \in \mathcal{P}_{coh}$, since

$$\mathbb{D}(R[1]) = R\mathcal{H}om(R, R)[2-1] \simeq R[1]$$

- In fact, $R[1] \simeq \mathcal{P}_{U,\mathcal{O}_U}$ (the unique simple w/ supp $= \mathbb{C}^2$), since $(SL_2)_{[1]} = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$ has a unique simple rep)
- Of course, R is not simple in $\mathrm{Mod}_R^{SL_2, fg}$, since $I = (x,y) \subset R$ is nontrivial subobject
- But while $I[1]$ is perverse, it is no longer a subobject: the exact triangle

$$I \longrightarrow R \longrightarrow S$$

rotates twice to become

$$S \longrightarrow I[1] \longrightarrow R[1].$$

$\mathcal{P}_{coh}$ is closed under extensions, hence $I[1] \in \mathcal{P}_{coh}$ and is now an <u>extension</u> of $R[1]$ rather than a subobject

- In particular, $S \subset I[1]$ is a composition series for $I[1]$, and $\{S, R[1]\}$ are the composition factors.