Main Exercise 1. We consider the subgroup $G = \text{Sp}_4$ of $\text{GL}_4$ of automorphisms preserving the symplectic form

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

More precisely, it is defined by

$$G = \{ M \in \text{GL}_4 \mid (t^* M) J M = J \}$$

(a) Show that $G$ is a linear algebraic group.

(b) Determine the set of diagonal matrices in $G$ and show that it is a maximal torus of $G$, which we will denote by $T$ (Hint: compute the centralizer of $T$).

(c) Let us consider the following matrices:

$$s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Show that $s$ and $t$ lie in $G$ and normalize $T$. Check that their image in $W = N_G(T)/T$ are generators of order 2 of $W$.

(d) Show that the following matrices form unipotent subgroups of $G$ when $x$ varies in $K$

$$\begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Finding the roots:

- Show that the conjugation by $T$ on each of the previous matrices is given by multiplication of $x$ by a multiplicative character of $T$ and express these characters on the basis of diagonal coordinates of $T$. We will denote by $\Phi^+$ the set form by these characters.

- Find a basis of $\Phi^+$ where all the elements are non-negative combination of the basis elements.

- Compute the index of the lattice $\mathbb{Z}\Phi^+$ in the lattice of characters of $T$.

(f) Determine the Levi subgroups of $G$ containing $T$. 

Exercise 1. We work with $G = \text{GL}_n$ and we take $T$ to be the maximal torus consisting of invertible diagonal matrices.

(a) Show that $T_S = T \cap \text{SL}_n$ (resp. $T_P = T/\mathbb{Z}(G)$) is a maximal torus of $\text{SL}_n$ (resp. $\text{PGL}_n$).

(b) Check that the corresponding Weyl groups are isomorphic.

Let $X(T) = \text{Hom}(T, G_m)$ be the group of characters of $T$.

(c) Determine the action of $W$ on $X(T)$ using the diagonal coordinates of $T$.

(d) Show that $T_S \hookrightarrow T$ and $T \twoheadrightarrow T_P$ induce linear maps $X(T) \rightarrow X(T_S)$ and $X(T_P) \hookrightarrow X(T)$.

Given $i \neq j$ and $x \in G_a$ we set $u_{ij}(x) = I_n + xE_{i,j}$ and $U_{i,j} = \text{Im} u_{i,j}$.

(e) Show that there exists $\alpha_{i,j} \in X(T)$ such that

$$\forall t \in T, \forall x \in G_a \quad tu_{i,j}(x)t^{-1} = u_{i,j}(\alpha_{i,j}(t)x).$$

Determine $\alpha_{i,j}$ explicitly.

(f) Show that $\Phi = \{\alpha_{i,j} \mid i \neq j\}$ is preserved under the linear maps in (d).

(g) Compute $X(T_S)/\mathbb{Z}\Phi$ and $X(T_P)/\mathbb{Z}\Phi$.

Exercise 2. Let $G$ be an affine algebraic group over $K$ and let $U \subseteq G$ be a subgroup of $G$ (not necessarily closed). Show that the following hold:

(a) the closure $\overline{U} \subseteq G$, in the Zariski topology, is a subgroup of $G$,

(b) if $U$ is abelian then so is $\overline{U}$.

Exercise 3. A matrix $A \in \text{Mat}_n(K)$ is called unipotent if all the eigenvalues of $A$ are equal to 1. Let $U_n(K) = \{A \in \text{Mat}_n(K) \mid A \text{ is unipotent}\}$.

Show that the following hold:

(a) $U_n(K)$ is a Zariski closed subset of $\text{Mat}_n(K)$,

(b) $U_n(K)$ is irreducible.

Let $J_n \in U_n(K)$ be a Jordan block of size $n \times n$ with every diagonal entry equal to 1. Furthermore, let $X \subseteq U_n(K)$ be the set of all matrices whose Jordan normal form is $J_n$. We call the elements of $X$ regular unipotent. Show that the following hold:

(c) $X$ is a Zariski open subset of $U_n(K)$,

(d) $\dim U_n(K) = n(n - 1)$.

(Hint: (b). Let $U_n(K) \subseteq \text{GL}_n(K)$ be the subgroup of uni-upper triangular matrices. Consider the map $\text{GL}_n(K) \times U_n(K) \rightarrow \text{Mat}_n(K); (P, A) \mapsto P^{-1}AP$. (c). One possibility is to write the condition $A \in X$ as a condition on the rank of $A$. (d). Consider the map $\text{GL}_n(K) \rightarrow \text{Mat}_n(K); P \mapsto P^{-1}J_nP$.)
Exercise 4. Assume $T$ is a torus and let $S \leq T$ be a closed subgroup. Show that $T/S$ is a torus.

Exercise 5. Assume $T$ is a torus of dimension $n \geq 0$.

(a) For any $d \geq 1$ show that the set $T_d = \{t \in T \mid t^d = 1\}$ is a finite subset of $T$.

(b) Show that the set $\bigcup_{d \geq 1} T_d \subseteq T$ is dense in $T$.

(c) Let $G$ be a linear algebraic group and $U \leq G$ be a closed subgroup which is an $n$-dimensional torus. Show that if $G$ is connected and $U$ is normal in $G$ then $U \leq Z(G)$, where $Z(G)$ is the centre of $G$. 