WARTHOG 2018, Lecture I-1

Main Exercise 1. We consider the subgroup $\mathbf{G} = \mathrm{Sp}_4$ of GL_4 of automorphisms preserving the symplectic form

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

More precisely, it is defined by

$$\mathbf{G} = \{M \in \mathrm{GL}_4 \mid ({}^{\mathrm{tr}}M)JM = J\}$$

(a) Show that $\mathbf{G}$ is a linear algebraic group.

(b) Determine the set of diagonal matrices in $\mathbf{G}$ and show that it is a maximal torus of $\mathbf{G}$, which we will denote by $\mathbf{T}$ (Hint: compute the centralizer of $\mathbf{T}$).

(c) Let us consider the following matrices:

$$s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Show that s and t lie in $\mathbf{G}$ and normalize $\mathbf{T}$. Check that their image in $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ are generators of order 2 of W.

(d) Show that the following matrices form unipotent subgroups of $\mathbf{G}$ when x varies in K

$$\begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Finding the roots:

- Show that the conjugation by $\mathbf{T}$ on each of the previous matrices is given by multiplication of x by a multiplicative character of $\mathbf{T}$ and express these characters on the basis of diagonal coordinates of $\mathbf{T}$. We will denote by Φ^+ the set form by these characters.
- Find a basis of Φ^+ where all the elements are non-negative combination of the basis elements.
- Compute the index of the lattice $\mathbb{Z}\Phi^+$ in the lattice of characters of $\mathbf{T}$.

(f) Determine the Levi subgroups of $\mathbf{G}$ containing $\mathbf{T}$.

WARTHOG 2018, Lecture I-1 supplementary exercises

Exercise 1. We work with $\mathbf{G} = \mathrm{GL}_n$ and we take $\mathbf{T}$ to be the maximal torus consisting of invertible diagonal matrices.

(a) Show that $\mathbf{T}_\mathrm{S} = \mathbf{T} \cap \mathrm{SL}_n$ (resp. $\mathbf{T}_\mathrm{P} = \mathbf{T}/Z(\mathbf{G})$) is a maximal torus of SL_n (resp. PGL_n).

(b) Check that the corresponding Weyl groups are isomorphic.

Let $X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ be the group of characters of $\mathbf{T}$.

(c) Determine the action of W on $X(\mathbf{T})$ using the diagonal coordinates of $\mathbf{T}$.

(d) Show that $\mathbf{T}_\mathrm{S} \hookrightarrow \mathbf{T}$ and $\mathbf{T} \twoheadrightarrow \mathbf{T}_\mathrm{P}$ induce linear maps $X(\mathbf{T}) \twoheadrightarrow X(\mathbf{T}_\mathrm{S})$ and $X(\mathbf{T}_\mathrm{P}) \hookrightarrow X(\mathbf{T})$.

Given $i \neq j$ and $x \in \mathbb{G}_a$ we set $u_{ij}(x) = \mathrm{I}_n + xE_{i,j}$ and $\mathbf{U}_{i,j} = \mathrm{Im}\, u_{i,j}$.

(e) Show that there exists $\alpha_{i,j} \in X(\mathbf{T})$ such that

$$\forall t \in \mathbf{T}, \forall x \in \mathbb{G}_a \quad tu_{i,j}(x)t^{-1} = u_{i,j}(\alpha_{i,j}(t)x).$$

Determine $\alpha_{i,j}$ explicitly.

(f) Show that $\Phi = \{\alpha_{i,j} \mid i \neq j\}$ is preserved under the linear maps in (d).

(g) Compute $X(\mathbf{T}_\mathrm{S})/\mathbb{Z}\Phi$ and $X(\mathbf{T}_\mathrm{P})/\mathbb{Z}\Phi$.

Exercise 2. Let $\mathbf{G}$ be an affine algebraic group over K and let $\mathbf{U} \subseteq \mathbf{G}$ be a subgroup of $\mathbf{G}$ (not necessarily closed). Show that the following hold:

(a) the closure $\overline{\mathbf{U}} \subseteq \mathbf{G}$, in the Zariski topology, is a subgroup of $\mathbf{G}$,

(b) if $\mathbf{U}$ is abelian then so is $\overline{\mathbf{U}}$.

Exercise 3. A matrix $A \in \mathrm{Mat}_n(K)$ is called *unipotent* if all the eigenvalues of A are equal to 1. Let

$$\mathscr{U}_n(K) = \{A \in \mathrm{Mat}_n(K) \mid A \text{ is unipotent}\}.$$

Show that the following hold:

(a) $\mathscr{U}_n(K)$ is a Zariski closed subset of $\mathrm{Mat}_n(K)$,

(b) $\mathscr{U}_n(K)$ is irreducible.

Let $J_n \in \mathscr{U}_n(K)$ be a Jordan block of size $n \times n$ with every diagonal entry equal to 1. Furthermore, let $X \subseteq \mathscr{U}_n(K)$ be the set of all matrices whose Jordan normal form is J_n. We call the elements of X *regular unipotent*. Show that the following hold:

(c) X is a Zariski open subset of $\mathscr{U}_n(K)$,

(d) $\dim \mathscr{U}_n(K) = n(n-1)$.

(Hint: (b). Let $\mathrm{U}_n(K) \leqslant \mathrm{GL}_n(K)$ be the subgroup of uni-upper triangular matrices. Consider the map $\mathrm{GL}_n(K) \times \mathrm{U}_n(K) \to \mathrm{Mat}_n(K); (P, A) \mapsto P^{-1}AP$. (c). One possibility is to write the condition $A \in X$ as a condition on the rank of A. (d). Consider the map $\mathrm{GL}_n(K) \to \mathrm{Mat}_n(K); P \mapsto P^{-1}J_nP$.)

Exercise 4. Assume $\mathbf{T}$ is a torus and let $\mathbf{S} \leqslant \mathbf{T}$ be a closed subgroup. Show that $\mathbf{T}/\mathbf{S}$ is a torus.

Exercise 5. Assume $\mathbf{T}$ is a torus of dimension $n \geqslant 0$.

(a) For any $d \geqslant 1$ show that the set $\mathbf{T}_d = \{t \in \mathbf{T} \mid t^d = 1\}$ is a finite subset of $\mathbf{T}$.

(b) Show that the set $\bigcup_{d \geqslant 1} \mathbf{T}_d \subseteq \mathbf{T}$ is dense in $\mathbf{T}$.

(c) Let $\mathbf{G}$ be a linear algebraic group and $\mathbf{U} \leqslant \mathbf{G}$ be a closed subgroup which is an n-dimensional torus. Show that if $\mathbf{G}$ is connected and $\mathbf{U}$ is normal in $\mathbf{G}$ then $\mathbf{U} \leqslant Z(\mathbf{G})$, where $Z(\mathbf{G})$ is the centre of $\mathbf{G}$.