WARTHOG 2018, Lecture I-2

Main Exercise 1. We work in the standard setup, but without assuming that F acts trivially on W. Given $\mathbf{S}$ a torus, we denote by $X(\mathbf{S}) = \hom(\mathbf{S}, \mathbb{G}_m)$ the lattice of characters of $\mathbf{S}$.

(a) Assume that $\mathbf{S}$ is an F-stable torus. Show that there is a short exact sequence of groups

$$\begin{array}{ccccccc} 1 \to \mathbf{S}^F \to & \mathbf{S} & \to & \mathbf{S} & \to 1. \\ & s & \mapsto & sF(s)^{-1} & \end{array}$$

We assume that it induces a short exact sequence of abelian groups

$$0 \to X(\mathbf{S}) \stackrel{\mathrm{Id}-F}{\to} X(\mathbf{S}) \to X(\mathbf{S}^F) \to 0.$$

Deduce that

$$|\mathbf{S}^F| = |\det(\mathrm{Id} - F \mid X(\mathbf{S}))|.$$

(b) Let $w \in W$. Show that $wF : t \to wF(t)w^{-1}$ is a Frobenius endomorphism of $\mathbf{T}$.

(c) Application: compute the order of the finite tori $\mathbf{T}^{wF}$ of $\mathrm{Sp}_4(q)$.

(d) Assume now that $\mathbf{S}$ is an F-stable *maximal* torus. Show that there exists $g \in \mathbf{G}$ such that

- $\mathbf{S} = {}^g\mathbf{T}$;
- $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T})$.

The class w of $g^{-1}F(g)$ in W is called a *type* of $\mathbf{S}$.

(e) Show that if w and w' are types of $\mathbf{S}$ then w and w' are F-conjugate in W (this means that there is $v \in W$ such that $w' = v^{-1}wF(v)$).

(f) Show that two maximal tori are conjugate under $\mathbf{G}^F$ if and only if their types are F-conjugate.

(g) Given $\mathbf{S}$ a torus of type w, show that $(\mathbf{S}, F)$ is conjugate to $(\mathbf{T}, wF)$. In particular $\mathbf{S}^F \simeq \mathbf{T}^{wF}$.

(h) Deduce that $|\mathbf{S}^F| = \det(\mathrm{Id} - wF \mid X(\mathbf{T}))$.

WARTHOG 2018, Lecture I-2 supplementary exercises

Exercise 1. Let $\mathbf{G}$ be a linear algebraic group and $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$.

(a) Show that if $\mathbf{H}$ is connected then $(\mathbf{G}/\mathbf{H})^F = \mathbf{G}^F/\mathbf{H}^F$.

(b) What happens when $\mathbf{H}$ is not connected?

Exercise 2. Let $\mathbf{G}$ be a reductive group with Frobenius endomorphism F. Given $r \geqslant 1$, we can form the reductive group $\mathbf{G}^r$ with Frobenius F_r

$$F_r(g_1, \dots, g_r) = (F(g_2), F(g_3), \dots, F(g_r), F(g_1)).$$

(a) Show that F_r is a Frobenius endomorphism of $\mathbf{G}^r$.

(b) Show that the first projection induces an isomorphism $(\mathbf{G}^r)^{F_r} \xrightarrow{\sim} \mathbf{G}^{F^r}$.

Exercise 3. Let $\mathbf{L}$ be a Levi subgroup of GL_n such that $F(\mathbf{L}) = \mathbf{L}$ (with standard Frobenius). Determine $\mathbf{L}^F$ explicitly.

Let A be a finite group and $\phi \in \mathrm{Aut}(A)$ an automorphism of A then for any $x \in A$ we denote by $\mathcal{O}_{A,\phi}(x) = \{a^{-1}x\phi(a) \mid a \in A\}$ the ϕ-conjugacy class of x. Correspondingly we have the ϕ-centraliser $C_{A,\phi}(x) = \{a \in A \mid x = a^{-1}x\phi(a)\}$. Note that these are simply the orbits and stabilisers of the action $\cdot : A \times A \to A$ given by $a \cdot x = a^{-1}x\phi(a)$. We denote by $H^1(\phi, A)$ the set of all ϕ-conjugacy classes. Finally for any $a \in A$ we denote by $a\phi \in \mathrm{Aut}(A)$ the automorphism defined by $(a\phi)(x) = a\phi(x)a^{-1}$ for all $x \in A$.

Exercise 4. Assume A is abelian. Show that $|H^1(\phi, A)| = |A^\phi|$.

Exercise 5. Assume $a \in A$. Show that the following hold:

(a) $\mathcal{O}_{A,\phi}(xa) = \mathcal{O}_{A,a\phi}(x)a$,

(b) $C_{A,\phi}(xa) = C_{A,a\phi}(x)$.

Exercise 6. Show that we have a well-defined bijection

$$\begin{aligned} H^1(\phi, A) &\to H^1(\phi^{-1}, A) \\ \mathcal{O}_{A,\phi}(x) &\mapsto \mathcal{O}_{A,\phi^{-1}}(x^{-1}). \end{aligned}$$

Exercise 7. Assume that $\mathbf{G}$ acts on an algebraic variety X such that $F(g \cdot x) = F(g) \cdot F(x)$. Let $x \in X^F$. Show that there the map $g \cdot x \longmapsto g^{-1}F(g)$ induces a bijection between

(a) the $\mathbf{G}^F$-orbits in $(\mathbf{G} \cdot x)^F$;

(b) $H^1(F, A)$, where $A = \mathrm{Stab}_{\mathbf{G}}(x)/\mathrm{Stab}_{\mathbf{G}}(x)^\circ$.

Exercise 8. Use the previous exercise to show that there are three $\mathrm{SL}_2(q)$-conjugacy classes of unipotent elements in $\mathrm{SL}_2(q)$ (but only two $\mathrm{GL}_2(q)$-conjugacy classes).