WARTHOG 2018, Lecture I-3

Main Exercise 1. Let X be a finite set and $\mathbb{C}[X]$ (resp. $\mathbb{C}[X \times X]$) be the set of complex valued functions on X.

(a) Show that the map

$$\begin{array}{rcl} \mathbb{C}[X \times X] & \longrightarrow & \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X]) \\ f & \longmapsto & (x \mapsto \sum_{y \in X} f(x,y)y) \end{array} \qquad (1)$$

is an isomorphism of vector spaces.

(b) Observe that under this isomorphism, the algebra structure on $\mathbb{C}[X \times X]$ is given by convolution:

$$(f \star f')(x,z) = \sum_{y \in X} f(x,y)f'(y,z).$$

(c) What is the unit for the convolution?

(d) Let G be a finite group acting on X. We can consider the diagonal action of G on $X \times X$. Show that the isomorphism (1) restricts to an isomorphism

$$\mathbb{C}[X \times X]^G \simeq \mathrm{End}_G(\mathbb{C}[X]).$$

(e) Deduce the dimension of $\mathrm{End}_G(\mathbb{C}[X])$ as a $\mathbb{C}$-vector space.

We now consider the case where $G = \mathrm{SL}_2(q)$ and $X = \mathbb{P}_1(\mathbb{F}_q)$. Let h_1, $\underline{h}_s$, h_s be the characteristic functions of ΔX, $X \times X$ and $X \times X \smallsetminus \Delta X$ respectively.

(f) Express h_s in terms of $\underline{h}_s$ and h_1.

(g) Compute the convolution of h_1 with the other functions.

(h) Compute $\underline{h}_s \star \underline{h}_s$ and deduce $h_s \star h_s$.

(i) Find a non-trivial linear combination of h_s and h_1 which squares to 1. Deduce that $\mathbb{C}[X \times X]^G$ is isomorphic to the group algebra of $\mathbb{Z}/2\mathbb{Z}$.

Main Exercise 2. We work in the standard setup, but without assuming that F acts trivially on W. Given $s \in S$ we write s_F for the longest element in the parabolic subgroup of W generated by the orbit of s under F. Then $\widetilde{S} = \{s_F \mid s \in S\} \subseteq W^F$ makes $(W^F, \widetilde{S})$ a Coxeter system.

(a) Write explicitly s_F for (W, F) of type ${}^2A_2$, ${}^2A_3$ and ${}^3D_4$ and show that $(W^F, \widetilde{S})$ is a Coxeter system in each of this case. What is the type of W^F?

(b) Given $s \in S$ we set
$$q_s = \#(\mathbf{U} \cap {}^{s_F w_0}\mathbf{U})^F = [B : B \cap {}^{s_F}B].$$
Compute q_s for the finite reductive groups $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$.

(c) Recall that $e = |B|^{-1} \sum_{b \in B} b$. Let $w, w' \in W^F$ such that $\ell(ww') = \ell(w) + \ell(w')$. Show that $ewew'e = eww'e$.

(d) Let $\mathbf{G} = \mathrm{SL}_2$ with standard Frobenius F. Compute $esese$ in terms of ese and e

(e) We admit that the multiplication map
$$(\mathbf{U} \cap {}^{w}\mathbf{U}) \times (\mathbf{U} \cap {}^{ww_0}\mathbf{U}) \longrightarrow \mathbf{U}$$
is an isomorphism of varieties. Show that
$$Bs_FB = Bs_F(U \cap {}^{s_F w_0}U)$$
and deduce the formula for $esese$ in general.

(f) Let $\ell : W^F \to \mathbb{N}$ be the length function of W^F and for any $w \in W^F$ define
$$q_w = [B : {}^{\dot{w}}B \cap B] \quad \text{and} \quad h_w = q_w ewe.$$
Show that we have
$$h_s h_w = \begin{cases} h_{sw} & \text{if } \ell(sw) > \ell(w) \\ q_s h_{sw} + (q_s - 1)h_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$
for all $s \in \widetilde{S}$ and $w \in W^F$.

WARTHOG 2018, Lecture I-3 supplementary exercises

Now G denotes any finite group and $\mathbb{K}$ an algebraically closed field. If A is a $\mathbb{K}$-algebra then we will denote by A-**mod** the category of all finitely generated (left) A-modules.

Exercise 2. Let us denote by $\mathrm{Fun}(G)$ the set of all $\mathbb{K}$-valued functions $f : G \to \mathbb{K}$ on G (we do not assume that these respect the group structure). Given $f, g \in \mathrm{Fun}(G)$ we define their *convolution product* $f \cdot g \in \mathrm{Fun}(G)$ by setting

$$(f \cdot g)(x) = \sum_{y \in G} f(xy)g(y^{-1}).$$

for all $x \in G$. Check that this product endows $\mathrm{Fun}(G)$ with the structure of a $\mathbb{K}$-algebra. Furthermore, show that we have an isomorphism $\mathbb{K}G \cong \mathrm{Fun}(G)$ of $\mathbb{K}$-algebras.

Exercise 3. Let $e \in \mathbb{K}G$ be an idempotent then $e\mathbb{K}Ge \subseteq \mathbb{K}G$ is naturally a $\mathbb{K}$-algebra with identity e. Prove that we have an isomorphism of $\mathbb{K}$-algebras $\mathrm{End}_{\mathbb{K}G}(\mathbb{K}Ge)^{\mathrm{opp}} \cong e\mathbb{K}Ge$. (Hint: every $f \in \mathrm{End}_{\mathbb{K}G}(\mathbb{K}Ge)$ is uniquely determined by its value at e.)

We assume now that $\mathrm{char}(\mathbb{K}) = 0$. Let $H \leqslant G$ be a subgroup of G and consider the idempotent

$$e = \frac{1}{|H|} \sum_{h \in H} h \in \mathbb{K}H.$$

Consider the corresponding KG-module $\mathbb{K}Ge \cong \mathbb{K}G \otimes_{\mathbb{K}H} \mathbb{K}He \cong \mathbb{K}[G/H]$ which is nothing other than the module affording the induced character $\mathrm{Ind}_H^G(1_H)$. By Exercise 3 we see that to understand $\mathrm{End}_{\mathbb{K}G}(\mathbb{K}Ge)$ it is sufficient to understand the $\mathbb{K}$-algebra $\mathcal{H}(G, H) = e\mathbb{K}Ge$, which we call a *Hecke algebra*. The structure of this algebra is given by the following exercise.

Exercise 4. For any $x \in G$ we denote by $D_x \subseteq G$ the double coset HxH and by $T_x \in \mathbb{K}G$ the corresponding sum

$$T_x = \frac{1}{|H|} \sum_{g \in D_x} g.$$

Let $\mathcal{D} \subseteq G$ be a set of representatives for the double cosets $H\backslash G/H$. Prove that $\{T_x \mid x \in \mathcal{D}\}$ is a basis of $\mathcal{H}(G, H)$. We call this the *standard basis* of $\mathcal{H}(G, H)$. Furthermore, show that we have $T_xT_y = \sum_{z \in \mathcal{D}} \mu_{x,y,z}T_z$ for any $x, y \in \mathcal{D}$ where

$$\mu_{x,y,z} = [D_x \cap zD_y^{-1} : H].$$

Exercise 5. We consider the Hecke algebra $\mathcal{H} := \mathcal{H}_q(W)$ with equal parameters.

(a) We define $\tau : \mathcal{H} \to \mathbb{C}$ by $\tau(h_w) = \delta_{e,w}$. Show that

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}^\vee \\ h & \longmapsto & (h' \mapsto \tau(hh')) \end{array}$$

induces an isomorphism of $(\mathcal{H}, \mathcal{H})$-bimodules. We say that $\mathcal{H}$ is a *symmetric algebra* and that τ is a *symmetrizing form.*

Given $\phi \in \mathcal{H}^\vee$, we denote by $\phi^\vee$ the unique element of $\mathcal{H}$ whose image under the previous isomorphism is ϕ.

We now assume that $\mathcal{H}$ is split semisimple. Recall that every irreducible character χ of W yields an irreducible character χ_q of $\mathcal{H}$.

(b) Given χ an irreducible character of W, show that $\chi_q^\vee$ is a central element of $\mathcal{H}$ and that $\chi_q^\vee = \chi_q^\vee e_{\chi_q}$ where e_{χ_q} is the central idempotent attached to χ_q.

We define the Schur element S_χ to be the scalar on which $\chi_q^\vee$ acts on the simple representation associated to χ_q. More precisely

$$S_\chi = \omega_{\chi_q}(\chi_q^\vee)$$

where ω_{χ_q} is the central character associated to χ_q.

(c) Show that $\chi_q^\vee = S_\chi e_{\chi_q}$ (Hint: write how a central element decomposes on the basis of primitive central idempotents).

(d) Using the decomposition of $1_\mathcal{H}$ into central idempotents, show that

$$\tau = \sum_{\chi \in \operatorname{Irr} W} \frac{1}{S_\chi} \chi_q.$$

(e) Compute the Schur elements when $W = \mathfrak{S}_n$ with $n = 2, 3$.

(f) We assume that we are in the generic setup. Show that

$$\dim \rho_\chi = \frac{1}{S_\chi} \sum_{w \in W} q^{\ell(w)}.$$

(g) Application: compute the degrees of the principal series representations of $\mathrm{GL}_3(q)$.