WARTHOG 2018, Lecture I-4

Main Exercise 1. We work in the standard setup, with $\mathbf{G} = \mathrm{Sp}_4$.

(a) Compute the character table of W (it is a dihedral group of order 8).

(b) Give the list of the principal series unipotent characters.

(c) For each $\chi \in \operatorname{Irr} W$ we assume the following:

- $\dim \rho_\chi$ is a polynomial in $\mathbb{Q}[q]$ dividing $|\mathrm{Sp}_4(q)|$,
- $\rho_{1_W} = 1_{\mathrm{Sp}_4(q)}$ and $\rho_{\mathrm{sgn}} = \mathrm{St}_{\mathrm{Sp}_4(q)}$, a character of dimension q^4.
- $\dim \rho_{\chi|q=1} = \chi(1)$,
- the degree of the polynomial $\dim \rho_\chi$ for $\chi \neq 1, \mathrm{sgn}$ is 3.

Compute the dimension of each ρ_χ. (Note: $\dim \rho_\chi$ need not be in $\mathbb{Z}[q]$, just in $\mathbb{Q}[q]$.)

(d) Other method:

(i) Find the four 1-dimensional characters of the Hecke algebra $\mathcal{H}_q(W)$.

(ii) Show that

$$h_s \mapsto \begin{bmatrix} -1 & 0 \\ -q & q \end{bmatrix} \quad \text{and} \quad h_t \mapsto \begin{bmatrix} q & -2 \\ 0 & -1 \end{bmatrix}$$

defines a 2-dimensional representation of $\mathcal{H}_q(W)$.

(iii) Using the fact that $\mathrm{Trace}(h_w \mid \mathbb{C}G/B) = 0$ if $w \neq 1$ compute $\dim \rho_\chi$ for all $\chi \in \operatorname{Irr} W$.

WARTHOG 2018, Lecture I-4 supplementary exercises

Exercise 1. We consider the Hecke algebra $\mathcal{H} := \mathcal{H}_q(W)$ with equal parameters.

(a) We define $\tau : \mathcal{H} \to \mathbb{C}$ by $\tau(h_w) = \delta_{e,w}$. Show that

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}^\vee \\ h & \longmapsto & (h' \mapsto \tau(hh')) \end{array}$$

induces an isomorphism of $(\mathcal{H}, \mathcal{H})$-bimodules. We say that $\mathcal{H}$ is a *symmetric algebra* and that τ is a *symmetrizing form.*

Given $\phi \in \mathcal{H}^\vee$, we denote by $\phi^\vee$ the unique element of $\mathcal{H}$ whose image under the previous isomorphism is ϕ.

We now assume that $\mathcal{H}$ is split semisimple. Recall that every irreducible character χ of W yields an irreducible character χ_q of $\mathcal{H}$.

(b) Given χ an irreducible character of W, show that $\chi_q^\vee$ is a central element of $\mathcal{H}$ and that $\chi_q^\vee = \chi_q^\vee e_{\chi_q}$ where e_{χ_q} is the central idempotent attached to χ_q.

We define the Schur element S_χ to be the scalar on which $\chi_q^\vee$ acts on the simple representation associated to χ_q. More precisely

$$S_\chi = \omega_{\chi_q}(\chi_q^\vee)$$

where ω_{χ_q} is the central character associated to χ_q.

(c) Show that $\chi_q^\vee = S_\chi e_{\chi_q}$ (Hint: write how a central element decomposes on the basis of primitive central idempotents).

(d) Using the decomposition of $1_{\mathcal{H}}$ into central idempotents, show that

$$\tau = \sum_{\chi \in \operatorname{Irr} W} \frac{1}{S_\chi} \chi_q.$$

(e) Compute the Schur elements when $W = \mathfrak{S}_n$ with $n = 2, 3$.

(f) We assume that we are in the generic setup. Show that

$$\dim \rho_\chi = \frac{1}{S_\chi} \sum_{w \in W} q^{\ell(w)}.$$

(g) Application: compute the degrees of the principal series representations of $\mathrm{GL}_3(q)$.

Exercise 2. Assume M is a $\mathbb{K}G$-module and let us denote by E the opposite endomorphism algebra $\mathrm{End}_{\mathbb{K}G}(M)^{\mathrm{opp}}$ of M. Let us set

(a) $\mathfrak{F}_M(V) = \mathrm{Hom}_{\mathbb{K}G}(M, V)$ for any $V \in \mathbb{K}G$-**mod**,

(b) $\mathfrak{F}_M(\varphi) = (f \mapsto \varphi \circ f)$ for any $\varphi \in \mathrm{Hom}_{\mathbb{K}G}(V, V')$.

Check that this defines a functor $\mathfrak{F}_M : \mathbb{K}G$-**mod** $\to E$-**mod**.

Exercise 3 (The Fitting Correspondence)**.** Keep the notation of Exercise 2 and let us assume that M is semisimple. We denote by $\mathrm{Irr}(G \mid M)$ the set of simple $\mathbb{K}G$-modules, up to isomorphism, which occur as composition factors of M. More explicitly, let $M = M_1 \oplus \cdots \oplus M_k$ be a decomposition such that each M_j is a simple submodule. Let $I \subseteq \{1, ..., k\}$ be a subset such that each M_j is isomorphic to exactly one M_i with $i \in I$. Then $\mathrm{Irr}(G \mid M) = \{M_i \mid i \in I\}$. Prove that the following hold.

(a) E is a semisimple algebra

(b) $M_i \cong M_j$ as $\mathbb{K}G$-modules if and only if $\mathfrak{F}_M(M_i) \cong \mathfrak{F}_M(M_j)$ as E-modules for any $1 \leqslant i, j \leqslant k$.

(c) $\mathfrak{F}_M$ induces a bijection $\mathfrak{F}_M : \mathrm{Irr}(G \mid M) \to \mathrm{Irr}(E)$.

Deduce that, at the level of characters, we have a decomposition

$$\chi_M = \sum_{S \in \mathrm{Irr}(E)} \dim(S)\chi_{M,S},$$

where $\chi_{M,S} \in \mathrm{Irr}(G)$ is a character corresponding to S under (c). (Remark: the assumption that M is semisimple is satisfied if $\mathrm{char}(\mathbb{K}) = 0$. Also, the statements in (b) and (c) hold under much weaker assumptions on M.)