WARTHOG 2018, Lecture II-1

Main Exercise 1. This exercise is about the flag variety for $GL(E)$ when $E = \mathbb{C}^3$. Let $W = S_3$ with $s = (12)$ and $t = (23)$. We will write a flag V in E as

$$V = (L \subset P)$$

where L is a line and P is a plane. Similarly V' will be $(L' \subset P')$, etcetera.

(a) What are the 6 elements of W, and what are their reduced expressions and lengths?

(b) For each $w \in W$, what does it mean for $V \xrightarrow{w} V'$? You should only use statements like $L = L'$ or $P \neq P'$ or $L \not\subset P'$ or $L' \subset P$, etcetera.

(c) Here is a set of statements which you should not have found jointly in the above list: $L \neq L'$, $L \subset P'$, $L' \subset P$, $P \neq P'$. Are you worried?

(d) Justify the following statement: in the closure of an orbit, inequalities can change to equalities but not vice versa. For example, $L \neq L'$ can become $L = L'$, or $L \not\subset P'$ can become $L \subset P'$, but $L = L'$ can not become $L \neq L'$. Now use this to compute the Bruhat order on W.

Now consider the equation

$$V \xrightarrow{s} V' \xrightarrow{t} V'' \xrightarrow{s} V'''. \tag{1}$$

(e) Show that (1) implies that $V \xrightarrow{st} V''$ and $V \xrightarrow{sts} V'''$.

(f) Suppose that V and V'' are fixed. Show that there is a unique V' satisfying (1). Similarly, suppose that V and V''' are fixed. Show that there is a unique V' and V'' satisfying (1).

You have just computed that the fibers of

$$\mathcal{O}(s) \times_{\mathcal{B}} \mathcal{O}(t) \to \mathcal{O}(st) \quad \text{and} \quad \mathcal{O}(s) \times_{\mathcal{B}} \mathcal{O}(t) \times_{\mathcal{B}} \mathcal{O}(s) \to \mathcal{O}(sts)$$

are points, thus proving that these maps are isomorphisms. In the next exercise we will look at the map

$$\mu\colon \overline{\mathcal{O}}(s) \times_{\mathcal{B}} \overline{\mathcal{O}}(t) \times_{\mathcal{B}} \overline{\mathcal{O}}(s) \to \overline{\mathcal{O}}(sts) = \mathcal{B} \times \mathcal{B}.$$

Main Exercise 2. (Optional)

(a) What does it mean for (V, V') to be in $\overline{\mathcal{O}}(s)$? In $\overline{\mathcal{O}}(t)$?

(b) For $w = sts$, fix V and V''' with $V \xrightarrow{w} V'''$. Compute the set of all pairs (V', V'') such that $(V, V') \in \overline{\mathcal{O}}(s)$, $(V', V'') \in \overline{\mathcal{O}}(t)$, and $(V'', V''') \in \overline{\mathcal{O}}(s)$. (You are computing the fiber of μ over a point in the orbit $\mathcal{O}(w)$.) Repeat the exercise for $w = s$. Is there a difference?

WARTHOG 2018, Lecture II-1 supplementary exercises

Exercise 2. Let $E = \mathbb{C}^n$. Prove that $\mathcal{B}(E)$ is a $\mathbb{P}^1$-bundle whose base space is a $\mathbb{P}^2$-bundle whose base space is a $\cdots$ whose base space is a $\mathbb{P}^{n-2}$-bundle whose base space is $\mathbb{P}^{n-1}$.

Exercise 3. Follow the steps below to prove the Proposition from the lecture, that for any V and V', there exists a unique $w \in S_n$ and an ordered basis $(e_1, \ldots, e_n)$ of V such that $(e_{w(1)}, \ldots, e_{w(n)})$ is adapted to V'.

(a) Let $W_{ij} = (V_i \cap V'_j)/(V_i \cap V'_{j-1})$. Prove that the dimension of W_{ij} is either 0 or 1.

(b) Let $w(j)$ be the minimum value of i such that $\dim W_{ij} = 1$. Prove that w is a permutation.

(c) Similarly, let $W'_{ij} = (V_i \cap V'_j)/(V_{i-1} \cap V'_j)$, and $w'(i)$ be the minimum value of j such that $\dim W'_{ij} = 1$. Prove that w and w' are inverse permutations.

(d) Let e_i be any vector in $V_i \cap V'_{w'(i)}$ but not in V_{i-1}. Prove that the ordered basis $(e_1, \ldots, e_n)$ satisfies the desired properties.

Exercise 4. This exercise aims to prove in general that $\overline{BsB} = B \sqcup BsB$ inside G (which is equivalent to $\overline{\mathcal{O}}(s) = \mathcal{O}(s) \sqcup \mathcal{O}(1)$).

(a) Prove this by explicit computation in the case when $G = SL_2$.

(b) Let $P_s = B \sqcup BsB$. Argue that the closure $\overline{P_s}$ is a union of BwB for various $w \in W$. Now use a dimensional argument to deduce that $\overline{P_s} = P_s$.

(c) In fact, P_s is a group. Use this to argue that the image of the map $\mathcal{O}(s) \times_{\mathcal{B}} \mathcal{O}(s) \to \mathcal{B} \times \mathcal{B}$ lies within $\overline{\mathcal{O}}(s)$.

Exercise 5. This exercise explores the Bruhat order and its relationship to subsequences of reduced expressions. The exercise works in general type, but the unfamiliar are welcome to work in type A.

(a) Let $w = s_1 \cdots s_r$ be a reduced expression of $w \in W$. Show that the map

$$\mu\colon \overline{\mathcal{O}}(s_1) \times_{\mathcal{B}} \overline{\mathcal{O}}(s_2) \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \overline{\mathcal{O}}(s_r) \to \mathcal{B} \times \mathcal{B}$$

$$(B_1 \to B_2 \to \cdots \to B_{r+1}) \mapsto (B_1, B_{r+1})$$

is proper. (Hint: any map from a compact space is proper.) Fixing the flag B_{r+1}, show directly that the preimage $\mu^{-1}(-, B_{r+1})$ is compact.

(b) Show that the image of μ is $\overline{\mathcal{O}}(w)$. (Hint: what happens to a dense open set?)

(c) Using the fact that $\overline{\mathcal{O}}(s) = \mathcal{O}(1) \sqcup \mathcal{O}(s)$, deduce that the image of μ contains precisely those $\mathcal{O}(v)$ where v is obtained as a subexpression of $s_1 \cdots s_r$. (Note: If non-reduced subexpressions are bothering you, think about that later, don't worry about it for now. It has to do with part (c) of the previous exercise.)

(d) Which elements are smaller in the Bruhat order than the n-cycle $(123 \cdots n)$ in S_n?

Exercise 6. Let $I \subset S$ and W_I be the corresponding parabolic subgroup. Show that $\mathbf{B}W_I\mathbf{B}$ is a group, equal to the standard parabolic subgroup $\mathbf{P}_I$.

Exercise 7. An element $w \in W$ is said to be I-reduced if w has minimal length in the coset $W_I w$.

(a) Given $w \in W$, show that $W_I w$ contains a unique I-reduced element.

(b) Let w be I-reduced. Show that $\ell(vw) = \ell(v) + \ell(w)$ for all $v \in W_I$.

(c) Deduce that for any $w \in W$ we have

$$\mathbf{P}_I w \mathbf{B} = \mathbf{B} W_I w \mathbf{B}.$$

(d) Determine the decomposition of $\mathbf{G}/\mathbf{P}_I$ into $\mathbf{B}$-orbits.

(e) Application: $\mathbf{G} = \mathrm{GL}_n$ and $I = \{1, \ldots, n-1\}$. Show that $\mathbf{G}/\mathbf{P}_I \simeq \mathbb{P}_{n-1}$ and write explicitly the decomposition into $\mathbf{B}$-orbits.

Exercise 8. Let I, J be subsets of S.

(a) Let $w \in W$. Show that in $W_I w W_J$ there is a unique element v such that v is I-reduced and v^{-1} is J-reduced. (hint: take the unique element of minimal length). We will say that v is I-reduced-J.

(b) In general, the decomposition of $w \in W_I v W_J$ as a product xvy with $x \in W_I$ and $y \in W_J$ is not unique. Show that there is a unique decomposition which maximizes x and minimizes y in the Bruhat order. (Hint: What is the ambiguity in writing w as xvy?)

(c) Deduce the decomposition of $\mathbf{G}$ into double cosets for the action of $\mathbf{P}_I$ and $\mathbf{P}_J$.

The next few exercises explores the flag variety for the symplectic group $Sp(2n)$. Let $E = \mathbb{C}^{2n}$, equipped with a symplectic form ω (which is, in particular, nondegenerate). Recall that a subspace $V_i \subset E$ is called *isotropic* if ω restricts to V_i to be zero. If $\dim V_i = i$, this implies that $i \leqslant n$.

Exercise 9. (a) Show that any line is isotropic.

(b) Let $V_\bullet = (0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{2n} = E)$ be a complete flag in E, such that $V_i^\perp = V_{2n-i}$. The set of all such flags is the *symplectic flag variety* $\mathcal{B}(E, \omega)$. Prove that V_i is isotropic for any $i \leqslant n$. Conversely, prove that any partial flag $(0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset E)$ with each V_i isotropic gives rise to a unique complete flag as above.

(c) When writing matrices in $GL(E)$ let us choose the ordered basis $(x_1, \ldots, x_n, y_n, \ldots, y_1)$, where $\omega(x_i, y_j) = \delta_{ij}$. Then one has the standard symplectic flag

$$\mathrm{Std} = (0 \subset \langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots \subset \langle \mathbf{x} \rangle \subset \langle \mathbf{x}, y_n \rangle \subset \langle \mathbf{x}, y_n, y_{n-1} \rangle \subset \cdots \subset \langle \mathbf{x}, \mathbf{y} \rangle = E),$$

where $\mathbf{x}$ denotes the entire set $\{x_1, \ldots, x_n\}$. Prove that the stabilizer of the standard symplectic flag is the intersection of upper triangular matrices in $GL(2n)$ with $Sp(2n)$.

Exercise 10. For this problem we look at $Sp(4)$, where $n = 2$.

(a) Inside $Sp(4)$ we have elements $\widetilde{s}$ and $\widetilde{t}$, where

$$\widetilde{s}(x_1) = x_2, \quad \widetilde{s}(x_2) = x_1, \quad \widetilde{s}(y_2) = y_1, \quad \widetilde{s}(y_1) = y_2, \tag{11}$$

$$\widetilde{t}(x_1) = x_1, \quad \widetilde{t}(x_2) = y_2, \quad \widetilde{t}(y_2) = -x_2, \quad \widetilde{t}(y_1) = y_1. \tag{12}$$

These descend to the generators s and t of the Weyl group. Confirm that $(st)^4 = 1$, $s^2 = 1$ and $t^2 = 1$.

(b) We will write a symplectic flag as $V = (L \subset P \subset L^\perp)$ where P is isotropic. Compare Std with $s \cdot \mathrm{Std}$ and $t \cdot \mathrm{Std}$ (which you can compute by applying $\widetilde{s}$ and $\widetilde{t}$). Come up with a rule (using statements like $L = L'$, $P \neq P'$, etcetera) which describes what it means for $V \xrightarrow{s} V'$ and $V \xrightarrow{t} V'$.

(c) Using the isomorphism $\mathcal{O}(s) \times_{\mathcal{B}} \mathcal{O}(t) \to \mathcal{O}(st)$, etcetera, you now can deduce what it means for $V \xrightarrow{w} V'$ for any $w \in W$. (One can also do this using dimension counting, analogous to the numbers d_{ij}^w.) Since the longest element is $stst = tsts$, two symplectic flags in generic position if $V \xrightarrow{stst} V'$. What about the relative position $V \xrightarrow{sts} V'$ is not generic? What about the relative position $V \xrightarrow{tst} V'$ is not generic?

(d) Confirm that if V and V'''' are in generic position, and

$$V \xrightarrow{s} V' \xrightarrow{t} V'' \xrightarrow{s} V''' \xrightarrow{t} V'''',$$

then V', V'', and V''' are uniquely determined.

Exercise 13. This exercise is a repeat of the exercise for function lovers, in different language. Itexplores the Hecke algebra and its relation with the flag variety over a finite field, using the idea of magic hammers. A magic hammer on $\mathbb{P}^1$ is a hammer which strikes the line in the plane, spinning it around until it lands randomly at some other line, but (magically) it never returns to the same line it started at. (The remaining lines are hit with equal probability.) On the flag variety in 3 dimensions there are two kinds of magic hammer, one which spins the line within the plane (like an arrow being flicked on a game board) and one which spins the plane around the line (like the flag being blown in the breeze around a flagpole).

(a) How many points are there in $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{F}_q}$? (Hint: $q+1$. Why?)

(b) Let T be the magic hammer on $\mathbb{P}^1$, viewed as a probabilistic operator which sends each line to one of the other q lines with equal probability $\frac{1}{q}$. There is also the identity operator 1, which sends each line to itself with probability 1. Write T^2 as a linear combination of T and 1.

(c) Renormalize so that $h = qT$. This is no longer a probability operator (the possibilities do not add up to 1) but that's ok. Write h^2 as a linear combination of h and 1.

(d) Probability operators like T act on probability distributions, which are functions $\mathbb{P}^1 \to [0,1]$ for which the total sum is 1. More general operators like h act on the finite dimensional vector space of all functions $\mathbb{P}^1 \to \mathbb{R}$. For a subset $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ one also has an operator on functions $\mathbb{P}^1 \to \mathbb{R}$ which sends a function f to the function $X \star f$, where

$$X \star f(L) = \sum_{(L',L)\in X} f(L').$$

(This is a specific example of something called *convolution*.) Can we describe T or h as $X \star (-)$ for some X? If so, what is X? What about the identity operator 1?

(e) Now we consider operators on functions $\mathcal{B} \to \mathbb{R}$ where $\mathcal{B}$ is the flag variety of $\mathbb{F}_q^3$. Now we have two magic hammers T_s and T_t, renormalized to h_s and h_t. Why do these generate an action of the Hecke algebra? What space $X \subset \mathcal{B} \times \mathcal{B}$ corresponds to the operator $h_s h_t h_s$?