We work in the standard setup.

**Main Exercise 1.** We assume $G = SL_2$. We consider the following Borel subgroup and maximal torus:

$$B = \left\{ \begin{bmatrix} \lambda & * \\ \cdot & \lambda^{-1} \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} \lambda & \cdot \\ \cdot & \lambda^{-1} \end{bmatrix} \right\}$$

(a) Show that the maps

$$G \rightarrow \mathbb{A}^2 \setminus \{(0,0)\} \rightarrow \mathbb{P}_1$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \mapsto (a, b) \mapsto [a : b]$$

induce $G$-equivariant isomorphisms $G/U \rightarrow \mathbb{A}^2 \setminus \{(0,0)\}$ and $G/B \rightarrow \mathbb{P}_1$.

(b) Let $s = \begin{bmatrix} \cdot & 1 \\ -1 & \cdot \end{bmatrix}$. Describe the cosets $BsB$ and $UsU$ explicitly.

(c) Deduce that

$$\tilde{X}(s) \cong \{(x, y) \in \mathbb{A}_2 \mid xy^q - yx^q = 1\}$$

$$X(s) \cong \{[x : y] \in \mathbb{P}_1 \mid xy^q - yx^q \neq 0\}$$

with the natural map $\tilde{X}(s) \rightarrow X(s)$ being the quotient by $T^sF$.

(d) Show that the map

$$\tilde{X}(s) \rightarrow \mathbb{A}_1 \setminus \{0\}$$

$$(x, y) \mapsto y$$

induces an isomorphism $U/\tilde{X}(s) \rightarrow \mathbb{A}_1 \setminus \{0\}$. 
WARThOG 2018, Lecture II-2 supplementary exercises

Exercise 1. We assume $G = SL_2$. We use the description of $\tilde{X}(s)$ given in the the main exercise.

(a) Show that the map
$$\tilde{X}(s) \longrightarrow \mathbb{A}_1$$
$$(x,y) \longrightarrow xy^2 - yx^2$$
induces and isomorphism $SL_2(q) \tilde{X}(s) \sim \mathbb{A}_1$.

(b) Compute $\#\tilde{X}(s)^{tF}$ for any $t \in T^s F$.

Exercise 2. Let $P = LV$ be a parabolic subgroup $P$ of $G$. Assume that $L$ is $F$-stable and define
$$\tilde{X}_L^G = \{ gV \in G/V \mid g^{-1}F(g) \in VF(V) \}$$
$$X_L^G = \{ gP \in G/P \mid g^{-1}F(g) \in PF(P) \}.$$

(a) Show that $G$ acts by left multiplication on these varieties. Show that $L$ acts by right multiplication on $\tilde{X}_L^G$ and that the natural map $G/V \to G/P$ induces a $G$-equivariant isomorphism
$$\tilde{X}_L^G \sim X_L^G.$$

(b) Let $Q$ be a parabolic subgroup of $G$ contained in $P$ and $M$ be a $F$-stable Levi complement of $Q$ such that $M \subset L$. In particular, $M$ is a Levi complement of $Q \cap L$, a parabolic subgroup of $L$. Show that the product induces an isomorphism
$$\tilde{X}_L^G \times_L \tilde{X}_M^L \sim \tilde{X}_M^Q.$$

(c) Assume that $L$ is a torus. Show that there exists $w \in W$ such that $\tilde{X}_L^G \sim \tilde{X}(w)$ and $X_L^G \sim X(w)$.

(d) More generally, show that there exists $I \subset S$ and $w \in W$ satisfying

- $w$ is $I$-reduced-$F(I)$,
- $W_I = wW_{F(I)}w^{-1}$,

such that $(P, L, F)$ is conjugate to $(P_I, L_I, \hat{w}F)$.

(e) Deduce that the varieties $\tilde{X}_L^G$ and $X_L^G$ can be described by
$$\tilde{X}_L^G \simeq \{ gV \in G/U_I \mid g^{-1}F(g) \in U_I \hat{w} U_{F(I)} \}$$
$$X_L^G \simeq \{ gP \in G/P_I \mid g^{-1}F(g) \in P_{Iw} P_{F(I)} \}.$$