WARTHOG 2018, Lecture II-2

We work in the standard setup.

Main Exercise 1. We assume $\mathbf{G} = \mathrm{SL}_2$. We consider the following Borel subgroup and maximal torus:

$$\mathbf{B} = \left\{ \begin{bmatrix} \lambda & * \\ \cdot & \lambda^{-1} \end{bmatrix} \right\} \quad \text{and} \quad \mathbf{T} = \left\{ \begin{bmatrix} \lambda & \cdot \\ \cdot & \lambda^{-1} \end{bmatrix} \right\}$$

(a) Show that the maps

$$\begin{array}{ccccc} \mathbf{G} & \longrightarrow & \mathbb{A}^2 \smallsetminus \{(0,0)\} & \longrightarrow & \mathbb{P}_1 \\ \begin{bmatrix} a & c \\ b & d \end{bmatrix} & \longmapsto & (a,b) & \longmapsto & [a:b] \end{array}$$

induce $\mathbf{G}$-equivariant isomorphisms $\mathbf{G}/\mathbf{U} \xrightarrow{\sim} \mathbb{A}^2 \smallsetminus \{(0,0)\}$ and $\mathbf{G}/\mathbf{B} \xrightarrow{\sim} \mathbb{P}_1$.

(b) Let $s = \begin{bmatrix} \cdot & 1 \\ -1 & \cdot \end{bmatrix}$. Describe the cosets $\mathbf{B}s\mathbf{B}$ and $\mathbf{U}s\mathbf{U}$ explicitly.

(c) Deduce that

$$\begin{aligned} \widetilde{\mathbf{X}}(s) &\simeq \{(x,y) \in \mathbb{A}_2 \mid xy^q - yx^q = 1\} \\ \mathbf{X}(s) &\simeq \{[x:y] \in \mathbb{P}_1 \mid xy^q - yx^q \neq 0\} \end{aligned}$$

with the natural map $\widetilde{\mathbf{X}}(s) \twoheadrightarrow \mathbf{X}(s)$ being the quotient by $\mathbf{T}^{sF}$.

(d) Show that the map

$$\begin{array}{ccc} \widetilde{\mathbf{X}}(s) & \longrightarrow & \mathbb{A}_1 \smallsetminus \{0\} \\ (x,y) & \longmapsto & y \end{array}$$

induces and isomorphism $U \backslash \widetilde{\mathbf{X}}(s) \xrightarrow{\sim} \mathbb{A}_1 \smallsetminus \{0\}$.

WARTHOG 2018, Lecture II-2 supplementary exercises

Exercise 1. We assume $\mathbf{G} = \mathrm{SL}_2$. We use the description of $\widetilde{\mathbf{X}}(s)$ given in the the main exercise.

(a) Show that the map

$$\begin{array}{ccc} \widetilde{\mathbf{X}}(s) & \longrightarrow & \mathbb{A}_1 \\ (x,y) & \longmapsto & xy^{q^2} - yx^{q^2} \end{array}$$

induces and isomorphism $\mathrm{SL}_2(q)\backslash\widetilde{\mathbf{X}}(s) \xrightarrow{\sim} \mathbb{A}_1$.

(b) Compute $\#\widetilde{\mathbf{X}}(s)^{tF}$ for any $t \in \mathbf{T}^{sF}$.

Exercise 2. Let $\mathbf{P} = \mathbf{LV}$ be a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Assume that $\mathbf{L}$ is F-stable and define

$$\begin{aligned} \widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} &= \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V}F(\mathbf{V})\} \\ \mathbf{X}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} &= \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}. \end{aligned}$$

(a) Show that G acts by left multiplication on these varieties. Show that L acts by right multiplication on $\widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and that the natural map $\mathbf{G}/\mathbf{V} \to \mathbf{G}/\mathbf{P}$ induces a G-equivariant isomorphism

$$\widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}/L \xrightarrow{\sim} \mathbf{X}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}.$$

(b) Let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{G}$ contained in $\mathbf{P}$ and $\mathbf{M}$ be a F-stable Levi complement of $\mathbf{Q}$ such that $\mathbf{M} \subset \mathbf{L}$. In particular, $\mathbf{M}$ is a Levi complement of $\mathbf{Q} \cap \mathbf{L}$, a parabolic subgroup of $\mathbf{L}$. Show that the product induces an isomorphism

$$\widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \times_L \widetilde{\mathbf{X}}^{\mathbf{L}}_{\mathbf{M}\subset\mathbf{Q}\cap\mathbf{L}} \xrightarrow{\sim} \widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{M}\subset\mathbf{Q}}.$$

(c) Assume that $\mathbf{L}$ is a torus. Show that there exists $w \in W$ such that $\widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \simeq \widetilde{\mathbf{X}}(\dot{w})$ and $\mathbf{X}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} \simeq \mathbf{X}(w)$.

(d) More generally, show that there exists $I \subset S$ and $w \in W$ satisfying

- w is I-reduced-$F(I)$,
- $W_I = wW_{F(I)}w^{-1}$,

such that $(\mathbf{P}, \mathbf{L}, F)$ is conjugate to $(\mathbf{P}_I, \mathbf{L}_I, \dot{w}F)$.

(e) Deduce that the varieties $\widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and $\mathbf{X}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ can be described by

$$\begin{aligned} \widetilde{\mathbf{X}}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} &\simeq \{g\mathbf{V} \in \mathbf{G}/\mathbf{U}_I \mid g^{-1}F(g) \in \mathbf{U}_I\dot{w}\mathbf{U}_{F(I)}\} \\ \mathbf{X}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} &\simeq \{g\mathbf{P} \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}F(g) \in \mathbf{P}_I w\mathbf{P}_{F(I)})\}. \end{aligned}$$