WARTHOG 2018, Lecture II-3

We work in the standard setup.

Main Exercise 1. Recall that w_0 is the unique element of W of maximal length, and that w_0 is an involution.

(a) Let $s \in S$. Show that there exists $t \in S$ such that $sw_0 = w_0 t$.

(b) Let $\boldsymbol{\pi} = w_0 \cdot w_0$ in the braid monoid B_W^+. Show that

  (i) π is central in B_W^+.

  ((i) Every $w \in W$ divides $\boldsymbol{\pi}$ on the left and on the right.

Deduce that for every $b \in B_W$ there exists $n \geqslant 0$ such that $\pi^n b \in B_W^+$.

(c) Let $b \in B_W$. Assume that $C_{B_W}(b)$ is finitely generated in B_W. Show that there exist $b' \in B_W^+$ and $b_1, \ldots, b_m \in B_W^+$ such that

- $C_{B_W}(b) = C_{B_W}(b')$;
- each b_i divides b';
- $\{b_1, \ldots, b_m\}$ generate $C_{B_W}(b')$.

WARTHOG 2018, Lecture II-3 supplementary exercises

Exercise 1. Using the element $\boldsymbol{\pi}$ studied in the main exercise, show that two elements of B_W^+ have a common multiple.

Exercise 2. Let $w = vv'$ with $\ell(v) + \ell(v') = \ell(w)$. Recall that there is a natural morphism $D_v : \mathbf{X}(w) \to \mathbf{X}(v^{-1}wF(v))$. Show that it is bijective.

Exercise 3. Let F' be the endomorphism of $\mathbf{G}^r$ defined by

$$F'(g_1, \ldots, g_r) = (g_2, \ldots, g_r, F(g_1)).$$

(a) Show that some power of F' is a Frobenius endomorphism of $\mathbf{G}^r$. Such F' is a particular case of a Steinberg endomorphism. Show that the first projection induces an isomorphism $(\mathbf{G}^r)^{F'} \xrightarrow{\sim} \mathbf{G}^F$.

(b) Let $w_1, \ldots, w_r \in W$ and $w = (w_1, w_2, \ldots, w_r) \in W^r$. Show that there is a G-equivariant isomorphism of varieties

$$\mathbf{X}_{\mathbf{G},F}(w_1, \ldots, w_r) \simeq \mathbf{X}_{(\mathbf{G})^r,F'}(w).$$

Exercise 4. Given $\mathbf{w} \in B_W^+$ we will write w for its image in W. We say that $\mathbf{w} \in B_W^+$ is a *good d-th root of $\boldsymbol{\pi}$* if $\mathbf{w}^d = \boldsymbol{\pi}$ and if $\mathbf{w}^m$ is reduced for all $m \leqslant d/2$.

(a) Show the equivalence between:

- $\mathbf{w}$ is a good d-th root of $\boldsymbol{\pi}$;
- $w^d = 1$ and $\ell(w^m) = \ell(\boldsymbol{\pi})m/d$ for all $m \leqslant d/2$.

(Hint: start with the easy case where d is even)

(b) Find the good roots of π for $W = \mathfrak{S}_4$.

Exercise 5. Let $\mathbf{w}$ be a good d-th root of $\boldsymbol{\pi}$. Show that $\mathbf{X}(\mathbf{w})^{F^m} = \varnothing$ for all $1 \leqslant m < d$.

Exercise 6. Let $\mathbf{w}$ be a d-th root of $\boldsymbol{\pi}$. Show that the Deligne–Lusztig variety $\mathbf{X}(\mathbf{w})$ for $(\mathbf{G}, F)$ embeds naturally in the Deligne–Lusztig variety $\mathbf{X}(\boldsymbol{\pi})$ for $(\mathbf{G}, F^d)$.

Exercise 7. We assume $W = \mathfrak{S}_n$.

(a) All n-cycles in W are conjugate to $(1, \ldots, n)$. Show that the lift of an n-cycle to B_W^+ is conjugate to $(1, \ldots, n)$ in B_W^+ if and only if $\ell(w) = n - 1$. (Hint: use induction on n.)

(b) Find an n-cycle which is a good n-th root of π. Deduce that the n-th root of π are exactly the n-cycles with Coxeter length $n - 1$.