WARTHOG 2018, Lecture III-1

Main Exercise 1. Let X be a variety over $\mathbb{F}_q$, and let $U \subset X$ be an open subset. Recall that we have a long exact sequence in compactly supported ℓ-adic étale cohomology:

$$\cdots \to H^i_c(U) \to H^i_c(X) \to H^i_c(X \setminus U) \to H^{i+1}_c(U) \to \cdots$$

Furthermore, if X and U are both defined over $\mathbb{F}_q$, then the Frobenius endomorphism acts on all of these cohomology groups, and the sequence is Frobenius equivariant.

Recall also that $H^i_c(\mathbb{A}^n) = 0$ unless $i = 2n$, in which case it is one-dimensional and Fr_q acts as multiplication by q^n.

(a) Prove that the odd cohomology of $\mathbb{P}_n$ vanishes, and that Fr acts on $H^{2i}(\mathbb{P}_n) \simeq H^{2i}_c(\mathbb{P}^n)$ as multiplication by q^i. (Hint: Let $U = \mathbf{A}^n$.)

(b) A similar argument works for any variety admitting a stratification by affine spaces. In particular, it works for the flag variety $\mathbf{G}/\mathbf{B}$. Use this fact to give a formula for the Poincaré polynomial of $\mathbf{G}/\mathbf{B}$.

(c) Let $\mathcal{A}$ be a finite set of hyperplanes in $\mathbb{A}^n_{\mathbb{F}_q}$, and let $M_{\mathcal{A}}$ be the complement of the corresponding finite set of hyperplanes in $\mathbb{A}^n$. Prove that $H^{2n-i}_c(M_{\mathcal{A}}) = 0$ unless $0 \leqslant i \leqslant 2$ and Fr acts on $H^{2n-i}_c(M_{\mathcal{A}})$ as multiplication by q^{n-i}. (Hint: Define $\mathcal{A}'$ by deleting a hyperplane and $\mathcal{A}''$ by restricting to that hyperplane, so that $M_{\mathcal{A}''} \simeq M_{\mathcal{A}'} \setminus M_{\mathcal{A}}$.)

Exercise 1. (Supplemental) Let $w = vv'$ with $\ell(v)+\ell(v') = \ell(w)$. Recall that there is a natural morphism $D_v : \mathbf{X}(w) \to \mathbf{X}(v^{-1}wF(v))$. Show that it induces a G-equivariant isomorphism

$$H^\bullet_c\big(\mathbf{X}(w)\big) \xrightarrow{\sim} H^\bullet_c\big(\mathbf{X}(v^{-1}wF(v))\big).$$

(Hint: compute $D_{v'} \circ D_v$.)