WARTHOG 2018, Lecture III-2

Main Exercise 1. Let s be a positive integer prime to p. We consider the curve

$$X_s = \{(a, \lambda) \in \mathbb{A}_1 \times \mathbb{G}_m \mid a^q - a = \lambda^s\}.$$

(a) Observe that $\mathbb{F}_q$ (resp. the group of sth roots of unity μ_s) acts on X_s by translations on $a \in \mathbb{A}_1$ (resp. by multiplication on $\lambda \in \mathbb{G}_m$).

(b) Identify X_1 and computes its cohomology.

(c) Show that the map $(a, \lambda) \mapsto \lambda$ induces a μ_s-equivariant isomorphism $\mathbb{F}_q \backslash X_s \xrightarrow{\sim} \mathbb{G}_m$. Deduce $H_c^\bullet(X_s)^{\mathbb{F}_q}$.

(d) Show that the map $(a, \lambda) \mapsto (a, \lambda^s)$ induces an $\mathbb{F}_q$-equivariant isomorphism $\mu_s \backslash X_s \xrightarrow{\sim} X_1$. Deduce $H_c^\bullet(X_s)^{\mu_s}$.

(e) Let $\psi \in \operatorname{Irr} \mathbb{F}_q$ with $\psi \neq 1$. Using the property stated in Supplemental Exercise 1, show that $H_c^1(X_s)_\psi$ is a multiple of the regular representation of μ_s.

(f) Deduce the cohomology of X_s with the action of μ_s and $\mathbb{F}_q$.

WARTHOG 2018, Lecture III-2 supplementary exercises

Exercise 1. Let H be a finite group acting on a variety X. We admit that if $\mathrm{Stab}_H(x)$ is an ℓ'-group for every $x \in X$ then

$$h \longmapsto \sum (-1)^i \mathrm{Trace}(h \mid H_c^i(X, \mathbb{Q}_\ell)$$

is the character of a virtual projective $\mathbb{Z}_\ell H$-module (this is in particular the case if the action of H is free).

Show that for every p'-element $h \in H$ we have

$$\sum (-1)^i \mathrm{Trace}(h \mid H_c^i(X)) = \sum (-1)^i \dim H_c^i(X^h).$$

Exercise 2. Let $w \in W$ which we represent as $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$. Define

$$\mathbf{Y}(\dot{w}) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}w\mathbf{U}\}.$$

(a) Show that $\mathbf{G}^F$ acts freely by left multiplication on $\mathbf{Y}(\dot{w})$ and that $\mathbf{G}^F \backslash \mathbf{Y}(\dot{w}) \xrightarrow{\sim} \mathbf{U}\dot{w}\mathbf{U}$.

(b) Deduce the Euler characteristic of $\mathbf{Y}(\dot{w})$.

(c) Using the natural map $\mathbf{Y}(\dot{w}) \to \widetilde{\mathbf{X}}(\dot{w})$ induced by $\mathbf{G} \to \mathbf{G}/\mathbf{U}$, compute the Euler characteristic of $\widetilde{\mathbf{X}}(\dot{w})$.

(d) Deduce that the Euler characteristic of $\mathbf{X}(w)$ is

$$\chi_{\mathbf{X}(w)} = \frac{|\mathbf{G}^F|}{q^N |\mathbf{T}^{wF}|}$$

where $N = \ell(w_0) = \dim \mathbf{U}$.