Main Exercise 1. Let $s$ be a positive integer prime to $p$. We consider the curve

$$X_s = \{(a, \lambda) \in \mathbb{A}_1 \times \mathbb{G}_m \mid a^q - a = \lambda^s\}.$$

(a) Observe that $\mathbb{F}_q$ (resp. the group of $s$th roots of unity $\mu_s$) acts on $X_s$ by translations on $a \in \mathbb{A}_1$ (resp. by multiplication on $\lambda \in \mathbb{G}_m$).

(b) Identify $X_1$ and computes its cohomology.

(c) Show that the map $(a, \lambda) \mapsto \lambda$ induces a $\mu_s$-equivariant isomorphism $\mathbb{F}_q \backslash X_s \cong \mathbb{G}_m$. Deduce $H^\bullet_c(X_s)_{\mathbb{F}_q}$.

(d) Show that the map $(a, \lambda) \mapsto (a, \lambda^s)$ induces an $\mathbb{F}_q$-equivariant isomorphism $\mu_s \backslash X_s \cong X_1$. Deduce $H^\bullet_c(X_s)_{\mu_s}$.

(e) Let $\psi \in \text{Irr} \mathbb{F}_q$ with $\psi \neq 1$. Using the property stated in Supplemental Exercise 1, show that $H^1_c(X_s)_\psi$ is a multiple of the regular representation of $\mu_s$.

(f) Deduce the cohomology of $X_s$ with the action of $\mu_s$ and $\mathbb{F}_q$. 
WARTHOG 2018, Lecture III-2 supplementary exercises

**Exercise 1.** Let $H$ be a finite group acting on a variety $X$. We admit that if $\text{Stab}_H(x)$ is an $\ell'$-group for every $x \in X$ then

$$h \mapsto \sum (-1)^i \text{Trace}(h | H^i_c(X, \mathbb{Q}_\ell))$$

is the character of a virtual projective $\mathbb{Z}_\ell H$-module (this is in particular the case if the action of $H$ is free).

Show that for every $p'$-element $h \in H$ we have

$$\sum (-1)^i \text{Trace}(h | H^i_c(X)) = \sum (-1)^i \dim H^i_c(X^h).$$

**Exercise 2.** Let $w \in W$ which we represent as $\dot{w} \in N_G(T)$. Define

$$Y(\dot{w}) = \{g \in G \mid g^{-1} F(g) \in U\dot{w}U\}.$$ 

(a) Show that $G^F$ acts freely by left multiplication on $Y(\dot{w})$ and that $G^F \backslash Y(\dot{w}) \sim U\dot{w}U$.

(b) Deduce the Euler characteristic of $Y(\dot{w})$.

(c) Using the natural map $Y(\dot{w}) \to \tilde{X}(\dot{w})$ induced by $G \to G/U$, compute the Euler characteristic of $\tilde{X}(\dot{w})$.

(d) Deduce that the Euler characteristic of $X(w)$ is

$$\chi_{X(w)} = \frac{|G^F|}{q^N |T^w F|}$$

where $N = \ell(w_0) = \dim U$. 

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