WARTHOG 2018, Lecture IV-1

We now assume that $\mathbf{G}$ is either $\mathrm{GL}_2(K)$ or $\mathrm{SL}_2(K)$ and F is a split Frobenius endomorphism. We fix the following notation:

- $\mathbb{K}$ an algebraically closed field of characteristic 0.
- $\delta \in \{\pm 1\}$ such that $q \equiv \delta \pmod 4$ if q is odd and $\delta_- = -\delta$.
- $\lambda \in \mathbb{F}_{q^2}^\times$ an element of order $q^2 - 1$.
- $\sigma = \lambda^{q+1}$, resp., $\tau = \lambda^{q-1}$, an element of order $q-1$, resp., $q+1$, of $\mathbb{F}_{q^2}^\times$.
- $\kappa \in \mathbb{K}^\times$ a primitive (q^2-1)th primitive root of unity.
- $\varepsilon = \kappa^{q+1}$, resp., $\eta = \kappa^{q-1}$, a primitive $(q-1)$th, resp., $(q+1)$th, root of unity in $\mathbb{K}^\times$.
- $\xi \in \mathbb{F}_q^\times$ a non-square, i.e., $\xi \neq \mu^2$ for some $\mu \in \mathbb{F}_q^\times$.

We assume fixed a homomorphism $\mathbf{u} : \mathbb{G}_a \to \mathbf{G}$ such that

$$\mathbf{u}(c) = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

for all $c \in \mathbb{G}_a$. Moreover, if $\mathbf{G} = \mathrm{GL}_2(K)$ then we denote by $\mathbf{d} : \mathbb{G}_m \times \mathbb{G}_m \to \mathbf{G}$ the homomorphism defined by

$$\mathbf{d}(\alpha, \beta) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

Clearly we have $\mathbf{d}(\alpha, \beta) \in \mathrm{SL}_2(K)$ if and only if $\beta = \alpha^{-1}$. Moreover, for convenience, we denote by z the central element $\mathbf{d}(-1, -1)$.

Main Exercise 1. We work with $\mathbf{G} = \mathrm{SL}_2$ and the split torus

$$\mathbf{T} = \{\mathbf{d}(\alpha, \alpha^{-1}) \mid \alpha \in \mathbb{G}_m\} = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \middle| \alpha \in \mathbb{G}_m \right\}.$$

(a) Given $\theta \in \mathrm{Irr}\,\mathbf{T}^F$, decompose $R_e(\theta)$ in the following cases:

  (i) $\theta = 1_{\mathbf{T}^F}$;
  (ii) $\theta = {}^s\theta$ but $\theta \neq 1_{\mathbf{T}^F}$ (such θ is unique and we will denote it by θ_0);
  (iii) $\theta \neq {}^s\theta$.

(b) Give the number of irreducible characters obtained this way and compute their dimension. Identify these characters in Table 2.

(c) Given $\psi \in \mathrm{Irr}\,\mathbf{T}^{sF}$, decompose $R_s(\psi)$ in the following cases:

  (i) $\psi = 1_{\mathbf{T}^{sF}}$;
  (ii) $\psi = {}^s\psi$ but $\psi \neq 1_{\mathbf{T}^{sF}}$ (such ψ is unique and we will denote it by ψ_0);
  (iii) $\psi \neq {}^s\psi$.

(d) Assuming that $\dim R_s(\psi)$ does not depend on ψ give the number of irreducible characters obtained this way and compute their dimension.

(e) Conclude and check that $|\mathrm{SL}_2(q)| = \sum_{\chi \in \mathrm{Irr}\,\mathrm{SL}_2(q)} \chi(1)^2$.

Table 1: Character Table of $\mathrm{GL}_2(q)$.

	Class:	$\mathbf{d}(\sigma^a,\sigma^a)$	$\mathbf{d}(\sigma^a,\sigma^a)\mathbf{u}(1)$	$\mathbf{d}(\sigma^a,\sigma^b)$	$\mathbf{d}(\lambda^b,\lambda^{qb})$
	Condition:	$-$	$-$	$\sigma^a \neq \sigma^b$	$\lambda^b \neq \lambda^{qb}$
	Number:	$q-1$	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$
	Size:	1	q^2-1	$q(q+1)$	$q(q-1)$
$1_{G,i}$		ε^{2ai}	ε^{2ai}	$\varepsilon^{(a+b)i}$	ε^{bi}
$\mathrm{St}_{G,i}$		$q\varepsilon^{2ai}$	0	$\varepsilon^{(a+b)i}$	$-\varepsilon^{bi}$
$\widetilde{\rho}_{i,j}$	$(\varepsilon^i \neq \varepsilon^j)$	$(q+1)\varepsilon^{a(i+j)}$	$\varepsilon^{a(i+j)}$	$\varepsilon^{ai+bj}+\varepsilon^{bi+aj}$	0
$\widetilde{\pi}_k$	$(\eta^k \neq 1)$	$(q-1)\varepsilon^{ak}$	$-\varepsilon^{ak}$	0	$-\kappa^{bk}-\kappa^{qbk}$

There are $\frac{(q-1)(q-2)}{2}$ distinct characters $\widetilde{\rho}_{i,j}$ and $\frac{q(q-1)}{2}$ distinct characters $\widetilde{\pi}_k$.

Table 2: Character Table of $\mathrm{SL}_2(q)$ with q odd.

	Class:	z^c	$z^c\mathbf{u}(1)$	$z^c\mathbf{u}(\xi)$	$\mathbf{d}(\sigma^a,\sigma^{-a})$	$\mathbf{d}(\tau^b,\tau^{-b})$
	Condition:	$-$	$-$	$-$	$\sigma^a \neq \pm 1$	$\tau^b \neq \pm 1$
	Number:	2	2	2	$\frac{q-3}{2}$	$\frac{q-1}{2}$
	Size:	1	$\frac{q^2-1}{2}$	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$
1_G		1	1	1	1	1
St_G		q	0	0	1	-1
ρ_i	$(\varepsilon^{2i} \neq 1)$	$(q+1)\varepsilon^{ci}$	1	1	$\varepsilon^{ai}+\varepsilon^{-ai}$	0
π_j	$(\eta^{2j} \neq 1)$	$(q-1)\eta^{cj}$	-1	-1	0	$-\eta^{bj}-\eta^{-bj}$
ρ_0'		$\frac{q+1}{2}\delta^c$	$\frac{1}{2}(1+\sqrt{\delta q})\delta^c$	$\frac{1}{2}(1-\sqrt{\delta q})\delta^c$	$(-1)^a$	0
ρ_0''		$\frac{q+1}{2}\delta^c$	$\frac{1}{2}(1-\sqrt{\delta q})\delta^c$	$\frac{1}{2}(1+\sqrt{\delta q})\delta^c$	$(-1)^a$	0
π_0'		$\frac{q-1}{2}\delta_-^c$	$\frac{1}{2}(-1+\sqrt{\delta q})\delta_-^c$	$\frac{1}{2}(-1-\sqrt{\delta q})\delta_-^c$	0	$-(-1)^b$
π_0''		$\frac{q-1}{2}\delta_-^c$	$\frac{1}{2}(-1-\sqrt{\delta q})\delta_-^c$	$\frac{1}{2}(-1+\sqrt{\delta q})\delta_-^c$	0	$-(-1)^b$

There are $\frac{q-3}{2}$ distinct characters ρ_i and $\frac{q-1}{2}$ distinct characters π_j. Moreover $c \in \{0,1\}$.

Table 3: Character Table of $\mathrm{SL}_2(q)$ with q even.

	Class:	I	$\mathbf{u}(1)$	$\mathbf{d}(\sigma^a,\sigma^{-a})$	$\mathbf{d}(\tau^b,\tau^{-b})$
	Condition:	$-$	$-$	$\sigma^a \neq 1$	$\tau^b \neq 1$
	Number:	1	1	$\frac{q}{2}-1$	$\frac{q}{2}$
	Size:	1	q^2-1	$q(q+1)$	$q(q-1)$
1_G		1	1	1	1
St_G		q	0	1	-1
ρ_i	$(\varepsilon^i \neq 1)$	$q+1$	1	$\varepsilon^{ai}+\varepsilon^{-ai}$	0
π_j	$(\eta^j \neq 1)$	$q-1$	-1	0	$-\eta^{bj}-\eta^{-bj}$

There are $\frac{q}{2}-1$ distinct characters ρ_i and $\frac{q}{2}$ distinct characters π_j.

WARTHOG 2018, Lecture IV-1 supplementary exercises

Exercise 1. Verify the information concerning the conjugacy classes contained in Tables 1 to 3.

Exercise 2. Write down explicitly a geometric conjugate of $\mathbf{d}(\lambda^b, \lambda^{qb})$, for $1 \leqslant b \leqslant q^2 - 1$, which is contained in $\mathrm{GL}_2(q)$.

Exercise 3. Using the fact that $\mathrm{St}_G = R_e(1_T) - 1_G$, and that $R_e(1_T) = \mathrm{Ind}_B^G(1_B)$ is a permutation character, compute directly the values of the Steinberg character and verify that

$$\mathrm{St}_G(x) = \begin{cases} \pm|C_G(x)|_p & \text{if } x \text{ is semisimple,} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4. Using Tables 1 to 3 write the Deligne–Lusztig characters as a linear combination of irreducible characters.

Exercise 5. Describe the effect of the restriction map $\mathrm{Res}_{\mathrm{SL}_2(q)}^{\mathrm{GL}_2(q)}$ on the irreducible characters and the Deligne–Lusztig characters.

Exercise 6. Assume $\mathbf{G} = \mathrm{SL}_2(K)$ and q is odd. Using Table 2 compute the values of the virtual characters

$$\begin{gathered} \rho_0' + \rho_0'' + \pi_0' + \pi_0'' \\ \rho_0' + \rho_0'' - \pi_0' - \pi_0'' \\ \rho_0' - \rho_0'' + \pi_0' - \pi_0'' \\ \rho_0' - \rho_0'' - \pi_0' + \pi_0''. \end{gathered}$$

Exercise 7. We work in the standard setup. Let $w \in W$ and $\theta \in \mathrm{Irr}\,\mathbf{T}^{wF}$. Using the result in the first supplementary exercise of Lecture III-3, show that

$$\dim R_w(\theta) = \dim R_w(1) = \frac{|G|}{q^N|\mathbf{T}^{wF}|}$$

(the last equality was proven in the second supplementary exercise of Lecture III-3).