WARTHOG 2018, Lecture IV-2

In this section we assume the standard setup. Recall that given any central function $\chi \in \mathrm{Cent}(W)$ we can associate the almost character

$$R_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w) R_w^{\mathbf{G}}$$

whose irreducible constituents are unipotent characters.

Main Exercise 1. We will work with $\mathbf{G} = \mathrm{Sp}_4$. The Weyl group W of $\mathbf{G}$ is dihedral group of order 8, with generating set $S = \{s, t\}$. We give here the induction table for the irreducible representations of $\langle s \rangle$ and $\langle t \rangle$ to W.

	$1_W = 2 \cdot -$	$1^2 \cdot -$	$- \cdot 2$	$1 \cdot 1$	$\mathrm{sgn}_W = - \cdot 1^2$
1_s	1	1	$\cdot$	1	$\cdot$
sgn_s	$\cdot$	$\cdot$	1	1	1
1_t	1	$\cdot$	1	1	$\cdot$
sgn_t	$\cdot$	1	$\cdot$	1	1

(a) Write the decomposition of R_e using the previous notation for irreducible characters of W.

(b) Using the induction table and the result of the second supplementary exercise of Lecture IV-2, compute the Deligne–Lusztig characters R_s and R_t.

(c) Using the orthogonality relations, show that there exist a unipotent character ρ not in the principal series such that

$$R_{st} = 1_G + \mathrm{St}_G - \rho_{1.1} + \rho.$$

(d) Determine R_{w_0} (Hint: use the fact that $R_{1_W} = 1_G$).

(e) Compute the almost characters R_χ for each $\chi \in \mathrm{Irr}\, W$ and deduce the families of unipotent characters.

(f) Show that the biggest family corresponds to the finite group $A = \mathbb{Z}/2\mathbb{Z}$ by writing the Fourier matrix associated to this group.

WARTHOG 2018, Lecture IV-2 supplementary exercises

Exercise 1. Show that $\chi \in \mathrm{Cent}(W) \mapsto R_\chi \in \mathrm{Cent}(G)$ is an isometry.

Exercise 2. Let χ (resp. ψ) be a central function on W_I (resp. W). Show that we have

$$R_{\mathbf{L}_I}^{\mathbf{G}}(R_\chi) = R_{\mathrm{Ind}_{W_I}^W \chi} \quad \text{and} \quad {}^*R_{\mathbf{L}_I}^{\mathbf{G}}(R_\psi) = R_{\mathrm{Res}_{W_I}^W \psi}$$

Exercise 3. Let $\mathbf{G} = \mathrm{GL}_n$ with standard Frobenius F. For any $\chi \in \mathrm{Irr}\,\mathfrak{S}_n$ show that the following hold:

(a) χ is a $\mathbb{Z}$-linear combination of characters $\mathrm{Ind}_{W_I}^W(1_{W_I})$,

(b) $\pm R_\chi$ is an irreducible character.

Can you give an expression for the sign?

Exercise 4. Assume $\mathbf{P} \leqslant \mathbf{G}$ is an F-stable parabolic subgroup with F-stable Levi complement $\mathbf{L} \leqslant \mathbf{P}$. Show that we have ${}^*R_{\mathbf{L}}^{\mathbf{G}}(\mathrm{St}_{\mathbf{G}^F}) = \mathrm{St}_{\mathbf{L}^F}$.

Exercise 5. Given $w \in W$ we define

$${}^*R_w : \mathrm{Cent}(G) \longrightarrow \mathrm{Cent}(\mathbf{T}^{wF})$$

as the adjoint of R_w for the usual inner product of central functions.

(a) Given $w, w' \in W$, show that

$${}^*R_{w'} \circ R_w = \sum_{\substack{x \in W \\ x^{-1}wF(x) = w'}} \mathrm{ad}\, x$$

where by definition $(\mathrm{ad}\, x)(\theta) = {}^x\theta$ for all $\theta \in \mathrm{Irr}\, \mathbf{T}^{wF}$.

Let reg_G be the character of the regular representation of G.

(b) Show that for all $t \in \mathbf{T}^{wF}$ we have

$${}^*R_w(\mathrm{reg}_G)(t) = \sum (-1)^i \mathrm{Trace}(t | H_c^i(\widetilde{\mathbf{X}}(w))).$$

(c) Using the property of the cohomology stated in the first supplementary exercise of Lecture III-2, show that ${}^*R_w(\mathrm{reg}_G)$ vanishes on all non-trivial elements of $\mathbf{T}^{wF}$.

(d) Use the last supplementary exercise of Lecture IV-1 to conclude that ${}^*R_w(\mathrm{reg}_G) = (\dim R_w)\mathrm{reg}_{\mathbf{T}^{wF}}$.

Exercise 6. We define the operator π_{uni} on $\mathrm{Cent}(G)$ by

$$\pi_{\mathrm{uni}} = \frac{1}{W} \sum_{w \in W} R_w \circ {}^*R_w.$$

(a) Show that π_{uni} is the orthogonal projection of central function to the space of uniform functions.

(b) Assuming that reg_G is uniform, show using the previous Exercise that

$$\mathrm{reg}_G = \frac{1}{W} \sum_{w \in W} (\dim R_w) R_w(\mathrm{reg}_{\mathbf{T}^{wF}}).$$

(c) We admit that $R_w(\theta)$ has no unipotent constituents unless θ is trivial. Deduce from the previous equality that

$$\sum_{\rho \text{ unipotent}} (\dim \rho)\rho = \frac{1}{W} \sum_{w\in W} (\dim R_w) R_w.$$

(d) Application: compute the dimension of the unipotent character ρ found in the main exercise of Lecture IV-2 (note that the dimensions of all the other unipotent characters were determined in the main exercise of Lecture I-4).