Main Exercise 1. Let $G = \text{Sp}_4$ with standard Frobenius $F$. Let $S = \{s, t\}$ be the set of simple reflections of $W$ (a dihedral group of order 8). Recall from the main exercise of Lecture IV-2 that

$$R_{st} = 1_G + St_G - \rho_{1,1} + \rho$$

where $\rho$ is a cuspidal unipotent character.

(a) Let $I = \{s\} \subset \{s, t\}$. Compute the cohomology of the variety $U_I \backslash X(st)$ with the action of $L_I$ and the eigenvalues of $F$.

(b) Deduce the cohomology of $X(st)$.

(c) We admit that $F$ has eigenvalues $-q$ on $\rho$. Using the trace formula, determine the dimension of each irreducible representation occurring in $H^*_c(X(st))$.

(d) Compute the endomorphism algebra $\text{End}_G(H^*_c(X(st)))$. What happens when $q = \sqrt{7}$?
Exercise 1. Show that every two Coxeter elements can be obtained from each other by a finite sequence of cyclic shifts. (Hint: use induction on $|S|$ by removing a reflection of $S$ which commutes with every other reflection but 1).

Exercise 2. Let $S = S' \sqcup S''$ be a decomposition of $S$ such that all elements of $S'$ (resp. $S''$) commute with each other.

(a) Show that such a decomposition always exists.

Let $c' = \prod_{s' \in S'} s'$, $c'' = \prod_{s'' \in S''} s''$ and $c = c'c''$. In particular $c$ is a Coxeter element of $W$.

(b) Assume that $h$ is odd. Show that $w_0 = c''c^{(h-1)/2} = c^{(h-1)/2}c'$.

(c) Assume that $h$ is even. Show that $w_0 = c^{h/2}$.

(d) Deduce that every Coxeter element lifts in $B^+_W$ to an $h$-th root of $\pi$.

Exercise 3. Let $c$ be a Coxeter element of $W$. Recall that $F$ has exactly $h$ eigenvalues on $H^*_c(X(c))$ and that the eigenspaces are mutually non-isomorphic irreducible representations of $G$.

(a) Show that $X(c)^{F^i} = \emptyset$ for $1 \leq i < h$. (Hint: use the fifth supplementary exercise of Lecture II-3 and the previous exercise.)

(b) Let $\lambda_1, \ldots, \lambda_h$ be the eigenvalues of $F$. Show that the dimension of the $\lambda_i$-eigenspace equals

$$(-1)^d \frac{|G|}{|T_u F|} \lambda_i^{-1} \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1}$$

where $d$ is the degree of the cohomology in which it occurs. (Hint: use the trace formula and the formula for the Euler characteristic proved in the second supplementary exercise of lecture III-2).

(c) Application: give the dimension of the unipotent characters of $GL_n(q)$ associated to the hook partitions.