WARTHOG 2018, Lecture IV-3

Main Exercise 1. Let $\mathbf{G} = \mathrm{Sp}_4$ with standard Frobenius F. Let $S = \{s, t\}$ be the set of simple reflections of W (a dihedral group of order 8). Recall from the main exercise of Lecture IV-2 that

$$R_{st} = 1_G + \mathrm{St}_G - \rho_{1\cdot 1} + \rho$$

where ρ is a *cuspidal* unipotent character.

(a) Let $I = \{s\} \subset \{s, t\}$. Compute the cohomology of the variety $U_I \backslash \mathbf{X}(st)$ with the action of L_I and the eigenvalues of F.

(b) Deduce the cohomology of $\mathbf{X}(st)$.

(c) We admit that F has eigenvalues $-q$ on ρ. Using the trace formula, determine the dimension of each irreducible representation occurring in $H_c^\bullet(\mathbf{X}(st))$.

(d) Compute the endomorphism algebra $\mathrm{End}_G\big(H_c^\bullet(\mathbf{X}(st))\big)$. What happens when $q = \sqrt[4]{1}$?

WARTHOG 2018, Lecture IV-3 supplementary exercises

Exercise 1. Show that every two Coxeter elements can be obtained from each other by a finite sequence of cyclic shifs. (Hint: use induction on $|S|$ by removing a reflection of S which commutes with every other reflection but 1).

Exercise 2. Let $S = S' \sqcup S''$ be a decomposition of S such that all elements of S' (resp. S'') commute with each other.

(a) Show that such a decomposition always exists.

Let $c' = \prod_{s' \in S'} s'$, $c'' = \prod_{s'' \in S''} s''$ and $c = c'c''$. In particular c is a Coxeter element of W.

(b) Assume that h is odd. Show that $w_0 = c''c^{(h-1)/2} = c^{(h-1)/2}c'$.

(c) Assume that h is even. Show that $w_0 = c^{h/2}$.

(d) Deduce that every Coxeter element lifts in B_W^+ to an h-th root of $\boldsymbol{\pi}$.

Exercise 3. Let c be a Coxeter element of W. Recall that F has exactly h eigenvalues on $H_c^\bullet(\mathbf{X}(c))$ and that the eigenspaces are mutually non-isomorphic irreducible representations of G.

(a) Show that $\mathbf{X}(c)^{F^i} = \varnothing$ for $1 \leqslant i < h$. (Hint: use the fifth supplementary exercise of Lecture II-3 and the previous exercise.)

(b) Let $\lambda_1, \ldots, \lambda_h$ be the eigenvalues of F. Show that the dimension of the λ_i-eigenspace equals

$$(-1)^d \frac{|G|}{|\mathbf{T}^{wF}|} \lambda_i^{-1} \prod_{j \neq i} (\lambda_i - \lambda_j)^{-1}$$

where d is the degree of the cohomology in which it occurs. (Hint: use the trace formula and the formula for the Euler characteristic proved in the second supplementary exercise of lecture III-2).

(c) Application: give the dimension of the unipotent characters of $\mathrm{GL}_n(q)$ associated to the hook partitions.