WARTHOG 2018, Lecture IV-4

Main Exercise 1. We work in the standard setup, assuming that $W^F = W$.

(a) Let $b \in B_W^+$ and $s \in S$. Recall that

$$\mathbf{X}(b,s,s) = \{(\mathbf{B}_1,\dots,\mathbf{B}_r) \mid \mathbf{B}_1 \overbrace{\to \cdots \to}^{b} \mathbf{B}_{r-1} \xrightarrow{s} \mathbf{B}_r \xrightarrow{s} F(\mathbf{B}_1)\}.$$

Show that elements of $\mathbf{X}(b,s,s)$ must satisfy $\mathbf{B}_{r-1} = F(\mathbf{B}_1)$ or $\mathbf{B}_{r-1} \xrightarrow{s} F(\mathbf{B}_1)$.

(b) Show that $X_f = \{(\mathbf{B}_i) \in \mathbf{X}(b,s,s) \mid \mathbf{B}_{r-1} = F(\mathbf{B}_1)\}$ is a closed subvariety of $\mathbf{X}(b,s,s)$ and that the map

$$(\mathbf{B}_1,\dots,\mathbf{B}_r) \longmapsto (\mathbf{B}_1,\dots,\mathbf{B}_{r-2})$$

induces a line bundle $\mathbf{X}_f \longrightarrow \mathbf{X}(b)$. Deduce the cohomology of $\mathbf{X}_f$.

(c) Let $\mathbf{X}_o = \mathbf{X}(b,s,s) \setminus \mathbf{X}_f$ and

$$\mathbf{X}'_o = \{(\mathbf{B}_1,\dots,\mathbf{B}_r) \mid \mathbf{B}_1 \overbrace{\to \cdots \to}^{b} \mathbf{B}_{r-1} \xrightarrow{s} F(\mathbf{B}_1) \text{ and } \mathbf{B}_r \xrightarrow{s} F(\mathbf{B}_1)\}.$$

(i) Show that

$$\begin{array}{ccc} \mathbf{X}'_o & \longrightarrow & \mathbf{X}(b,s) \\ (\mathbf{B}_1,\dots,\mathbf{B}_r) & \longmapsto & (\mathbf{B}_1,\dots,\mathbf{B}_{r-1}) \end{array}$$

is a line bundle.

(ii) Show that $\mathbf{X}_o$ is open in $\mathbf{X}'_o$ and that the complement is isomorphic to $\mathbf{X}(b,s)$.

(iii) Deduce that $\sum(-1)^i H_c^i(\mathbf{X}_o) = 0$ as a virtual character of G.

(d) Deduce that the virtual representations afforded by the cohomology of $\mathbf{X}(b,s,s)$ and $\mathbf{X}(b)$ are equal.

(e) Show that the virtual character $\sum(-1)^i H_c^i(\mathbf{X}(b))$ depends only on the image of b in W.

(f) Example: compare the individual cohomology groups of $\mathbf{X}(t)$ and $\mathbf{X}(s,s,t)$ where s,t are the simple reflections of the Weyl group of GL_3.

WARTHOG 2018, Lecture IV-4 supplementary exercises

Exercise 1. Let $\mathbf{G} = \mathrm{GL}_n$ with the standard Frobenius. We consider the Deligne–Lusztig varieties

$$\mathbf{X}_n = \mathbf{X}((1,2,\ldots,n)) \quad \text{and} \quad \mathbf{Y}_n = \mathbf{X}((n-1,n)(1,2,\ldots,n))$$

with the convention that $\mathbf{Y}_2 = \mathbf{X}(\boldsymbol{\pi})$ when $n = 2$. The variety $\mathbf{X}_n$ is a Coxeter variety.

We recall that the trivial representation 1_G (resp. the Steinberg representation St_G) occurs only in the cohomology group of $\mathbf{X}(w)$ of degree $2\ell(w)$ (resp. of degree $\ell(w)$).

(a) Determine the individual cohomology groups of $\mathbf{Y}_2$ together with the eigenvalues of F.

Let $I = \{s_1, \ldots, s_{n-2}\}$ so that $L_I \simeq \mathrm{GL}_{n-1}(q) \times \mathrm{GL}_1(q)$. We assume that there is an F-equivariant long exact sequence of L_I-modules

$$\cdots \to H_c^{i-2}(\mathbf{Y}_{n-1})(1) \oplus H_c^{i-1}(\mathbf{Y}_{n-1}) \to {}^*R_{\mathbf{L}_I}^{\mathbf{G}}\big(H_c^i(\mathbf{Y}_n)\big) \to H_c^{i-2}(\mathbf{X}_{n-1})(1) \to \cdots$$

(b) Determine $H_c^i(\mathbf{Y}_n)$ for $n = 3, 4, 5$.

(c) Observe that the only partitions associated to the unipotent charaters occurring in these cohomology groups have $(n-1)$-core equal to (1).

(d) Determine $H_c^i(\mathbf{Y}_n)$ for all n.

(e) Check the conjectures of the lecture notes on this variety.