WARTHOG 2018, Lecture V-3

Main Exercise 1. We assume $\mathbf{G} = \mathrm{SL}_2$, $\ell > 2$ and $\ell \mid q+1$. We admit that the principal block satisfies the following properties:

- The characters in the block are $\{1, \mathrm{St}\} \sqcup \{-R_s(\theta) \mid \theta \in \mathrm{Irr}\,(\mathbf{T}^{sF})_\ell \smallsetminus \{1_{\mathbf{T}^{sF}}\}\}$.
- There are two simple modules in the block, which we denote by k (trivial module) and S.
- The shape of P_k and P_S are as given in the main exercise of lecture V-2.
- The character of P_k is $1 + \mathrm{St}$, whereas the character of P_S is

$$\mathrm{St} - \frac{1}{2} \sum_{\mathrm{Irr}\,(\mathbf{T}^{sF})_\ell \smallsetminus \{1_{\mathbf{T}^{sF}}\}} R_s(\theta).$$

(a) Show that the projective cover of the trivial representation has indeed character $1 + \mathrm{St}$.

(b) Using the generalized eigenspace of F on the cohomology complex of $\widetilde{\mathbf{X}}(s)$, show that there is indeed a projective module with the same character as P_S.

(c) Show that the complex $R\Gamma_c(\widetilde{\mathbf{X}}(s), k)$, cut by the principal block, is isomorphic in $D^b(kG)$ to

$$0 \longrightarrow P_S \oplus P_S \longrightarrow P_k \longrightarrow 0$$

(d) Deduce the geometric version of Broué's conjecture.

WARTHOG 2018, Lecture V-4

Main Exercise 2. We assume $\mathbf{G} = \mathrm{Sp}_4$, $\ell > 4$ and $\ell \mid q^2+1$. Recall that the cohomology over $\overline{\mathbb{Q}}_\ell$ of the Coxeter variety is given by

$$\begin{array}{c|ccc} i & 2 & 3 & 4 \\ \hline H^i_c(\mathbf{X}(c)) & \mathrm{St} \oplus \rho & \rho_{1\cdot 1} & 1 \\ F & (1,-q) & q & q^2 \end{array}.$$

We admit that the projective indecomposable modules in the principal block have character $\chi_{\mathrm{exc}} + \rho$, $\chi_{\mathrm{exc}} + \mathrm{St}$, $\mathrm{St} + \rho_{1\cdot 1}$, $\rho_{1\cdot 1} + 1$.

Determine $R\Gamma_c(\mathbf{X}_\ell, k)$ and show that it is a tilting complex for the principal block.