1.1 REDUCTIVE GROUPS

1) Linear algebraic groups

All the algebraic varieties will be over an algebraically closed field $\bar{k} = k$

**Definition:** A linear algebraic group $G$ is an affine variety with a structure of a group st.

- the multiplication $G \times G \rightarrow G$
- the inverse $G \rightarrow G$

are morphisms of algebraic varieties.

**Example:** $G_a = (k, +)$ additive group $\hookrightarrow \left[ \begin{array}{cc} 1 & \ast \\ \ast & 1 \end{array} \right] \subseteq \text{GL}_2(k)$

$G_a = \text{Spec } k[t]$

"multiplication" is $k[t] \rightarrow k[t] \otimes k[t]$

$\quad t \mapsto t \otimes 1 + 1 \otimes t$

"inverse" is $k[t] \rightarrow k[t]$

$\quad t \mapsto -t$

* $G_m = (k^*, x)$ multiplicative group $\subseteq \text{GL}_n(k)$

$G_m = \text{Spec } k[t, t^{-1}]$ mult. $\quad t \mapsto t \otimes t$

inv. $\quad t \mapsto t^{-1}$

* $\text{GL}_n$ is an algebraic group
$\text{Gl}_n = \text{Spec } k[x_{ij}, \text{det}^{-1}]$ and mult. & inv. are polynomial

**Exercise:** write the mult. and inv. explicitly as morphisms of $k$-algebras $k[\text{Gl}_n] \to k[\text{Gl}_n] \otimes k[\text{Gl}_n]$ and $k[\text{Gl}_n] \to k[\text{Gl}_n]$

More generally every closed subgroup of $\text{Gl}_n$ is an algebraic gp. and the converse holds:

explanation of terminology $\text{(affine} \to \text{linear)}$

**Thm:** Any linear algebraic group is a closed subgroup of $\text{Gl}_n$ for some $n \geq 1$.

**Consequence:** as in $\text{Gl}_n$, there is a notion of semi-simple and unipotent (1-nilpotent) elements and a multiplicative Jordan decomposition (they do not depend on the embedding $G \hookrightarrow \text{Gl}_n$)

2) **Remarkable subgroups**

Let $G$ be a connected linear algebraic group

**Def:** a **torus** is a linear alg. gp. isomorphic to $(\text{G}_m)^r$

A torus of $G$ is a closed subgp of $G$ which is a torus.
**Thm:** Any two maximal tori of $G$ (maximal for the inclusion) are conjugate under $G$

**Ex:** if $T$ is a torus of $G\ell_n$ then the elements of $T$ are simultaneously diagonalizable

$\Rightarrow$ $T$ is conjugate to a subgroup of $\begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}$

**def:** a Borel subgroup of $G$ is a maximal closed connected solvable subgroup of $G$

**Thm:**
(i) Any two Borel subgroups are conjugate under $G$
(ii) Any max. torus $T$ of $G$ is contained in a Borel subgroup $B$ of $G$ and such pairs $T \leq B$ are conjugate under $G$.
(iii) If $B$ is a Borel subgroup, then $N_G(B) = B$

**Ex:** By the thm of Lie-Kolchin, the elt of a connected solvable subgroup of $G\ell_n$ are simultaneously triangulizable

$\Rightarrow$ Borel subgroups of $G\ell_n$ are conjugate to $\begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}$
3) Reductive groups \( G \) connected alg. gp

**Def.** The radical \( R(G) \) of \( G \) is the maximal closed connected solvable normal subgroup of \( G \). The unipotent radical \( R_u(G) \) of \( G \) is the max. closed connected normal subgroup of \( G \) containing only unipotent elements.

**Ex:** \( G = GL_n \), \( B = \left( \begin{array}{c} \ast \ \ast \\ \ast \end{array} \right) \), \( T = \left( \begin{array}{c} \ast \\ \ast \end{array} \right) \)

\[ R(G) = Z(G) \quad R(B) = B \quad R(T) = T \]

\[ R_u(G) = \{1\} \quad R_u(B) = \left( \begin{array}{c} 1 \\ \ast \end{array} \right) \quad R_u(T) = \{1\} \]

In addition, \( B \cong T \times R_u(B) \)

(This is general for connected solvable groups)

**Def.** \( G \) is semisimple if \( R(G) = \{1\} \)

\( G \) is reductive if \( R_u(G) = \{1\} \)

\( R_u(G) \subseteq R(G) \) therefore semisimple \( \Rightarrow \) reductive

\( G/R(G) \) is semisimple, \( G/R_u(G) \) is reductive.
4) **Classification**

Let $G$ be a connected reductive group.

**Thm:** If $T$ is a maximal torus of $G$ then $C_G(T) = T$.

**Def:** Given $T$ a maximal torus we define “the” Weyl group $W$ of $G$ by

$$W = N_G(T)/C_G(T) = N_G(T)/T$$

**Ex:** $G = \text{GL}_n$, $T = \left( \begin{array}{cc} * & \cdot \\ \cdot & * \end{array} \right)$, $N_G(T)/T \cong S_n$.

If $B \supseteq T$ is a Borel subgroup of $G$, we define

$$S = \{ w \in W \mid B U B w B \text{ is a subgroup of } G \}$$

the set of simple reflections of $W$.

**Prop:** $(W, S)$ is a Coxeter system.

**Ex:** $G = \text{GL}_n$, $w \in S \iff w = \left[ \begin{array}{ccc} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{array} \right]$ $(i, j) \in \Delta_n$.

$W$ is the same for $G$, $[G, G]$ or $G/Z(G)$:

$\text{GL}_n$, $\text{SL}_n$, $\text{PGL}_n$.

We need to add a dual pair of integral rep of $W$ (not dahlm) to classify connected reductive groups.
Classification of irreducible \( W \)

- \( A_n \) \( \text{GL}_n, SL_n, PGL_n \)
- \( B_n \) \( \text{SO}_{2n+1}, \text{Spin}_{2n+1} \)
- \( C_n \) \( \text{Sp}_{2n} \)
- \( D_n \) \( \text{SO}_{2n}, \text{Spin}_{2n} \)
- \( E_n \) \( n = 6, 7, 8 \)
- \( F_4 \)
- \( G_2 \)