I.1 REDUCTIVE GROUPS

1) Linear algebraic groups

All the algebraic varieties will be over an algebraically closed field $k = \bar{k}$

def: a linear algebraic group G is an affine variety with a structure of a group s.t
- the multiplication $G \times G \longrightarrow G$
- the inverse $G \longrightarrow G$

are morphisms of algebraic varieties.

Ex: * $\mathbb{G}_a = (k, +)$ additive group $\simeq \begin{bmatrix} 1 & * \\ \cdot & 1 \end{bmatrix} \subseteq GL_2(k)$

$\mathbb{G}_a = \operatorname{Spec} k[t]$

"multiplication" is $k[t] \longrightarrow k[t] \otimes k[t]$

$t \longmapsto t \otimes 1 + 1 \otimes t$

"inverse" is $k[t] \longrightarrow k[t]$

$t \longmapsto -t$

* $\mathbb{G}_m = (k^\times, \times)$ multiplicative group $\simeq GL_1(k)$

$\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$ mult: $t \longmapsto t \otimes t$

inv : $t \longmapsto t^{-1}$

* GL_n is an algebraic group

$GL_n = \mathrm{Spec}\, k[x_{ij}, \det^{-1}]$ and mult. & inv. are polynomial

<u>Exercise</u>: write the mult. and inv. explicitely as morphisms of k-algebras $k[GL_n] \longrightarrow k[GL_n] \otimes k[GL_n]$ and $k[GL_n] \longrightarrow k[GL_n]$

More generally every closed subgroup of GL_n is an algebraic gp. and the converse holds:

explains the terminology (affine $\Leftrightarrow$ linear)

Thm: Any linear algebraic group is a closed subgroup of GL_n for some $n \geq 1$.

<u>Consequence</u>: as in GL_n, there is a notion of semisimple and unipotent ($1+$nilpotent) elements and a multiplicative Jordan decomposition (these do not depend on the embedding $G \hookrightarrow GL_n$)

2) <u>Remarkable subgroups</u>

Let G be a connected linear algebraic group

def: a torus is a linear alg. gp isomorphic to $(\mathbb{G}_m)^r$

A torus of G is a closed subgp of G which is a torus.

Thm: Any two maximal tori of G (maximal for the inclusion) are conjugate under G

Ex: if T is a torus of GL_n then the elements of T are simultaneously diagonalizable

$\Rightarrow$ T is conjugate to a subgroup of max. torus $\begin{pmatrix} * & & & \cdot \\ & * & & \\ & & \ddots & \\ \cdot & & & * \end{pmatrix}$

def: a **Borel subgroup** of G is a maximal closed connected solvable subgroup of G

Thm: (i) Any two Borel subgroups are conjugate under G

(ii) Any max. torus T of G is contained in a Borel subgroup B of G and such pairs $T \subseteq B$ are conjugate under G.

(iii) If B is a Borel subgroup, then $N_G(B) = B$

Ex: By the thm of Lie-Kolchin, the elts of a connected solvable subgroup of GL_n are simultaneously triangularizable

$\Rightarrow$ Borel subgroups of GL_n are conjugate to $\begin{pmatrix} * & & * \\ & \ddots & \\ \cdot & & * \end{pmatrix}$

3) Reductive groups G connected alg. gp

def: • The radical $R(G)$ of G is the maximal closed connected solvable normal subgroup of G
• The unipotent radical $R_u(G)$ of G is the max. closed connected normal subgroup of G containing only unipotent elements

Ex: $G = GL_n$ $B = \begin{pmatrix} * & & * \\ & \ddots & \\ \cdot & & * \end{pmatrix}$ $T = \begin{pmatrix} * & & \cdot \\ & \ddots & \\ \cdot & & * \end{pmatrix}$

$R(G) = Z(G)$ $R(B) = B$ $R(T) = T$
$R_u(G) = \{1\}$ $R_u(B) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ \cdot & & 1 \end{pmatrix}$ $R_u(T) = \{1\}$

in addition $B \simeq T \ltimes R_u(B)$
(This is general for connected solvable groups)

def: • G is semisimple if $R(G) = \{1\}$
• G is reductive if $R_u(G) = \{1\}$

$R_u(G) \subseteq R(G)$ therefore semisimple $\Rightarrow$ reductive

$G/R(G)$ is semisimple, $G/R_u(G)$ is reductive.

4) Classification

Let G be a connected reductive group

Thm: If T is a max. torus of G then $C_G(T) = T$

def: Given T a max. torus we define "the" Weyl group of G by $W = N_G(T)/C_G(T) = N_G(T)/T$

Ex: $G = GL_n$ $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ $N_G(T)/T \simeq \mathfrak{S}_n$

If $B \supseteq T$ is a Borel subgroup of G, we define
$S = \{ w \in W \mid B \cup BwB \text{ is a subgroup of } G \}$
the set of simple reflections of W

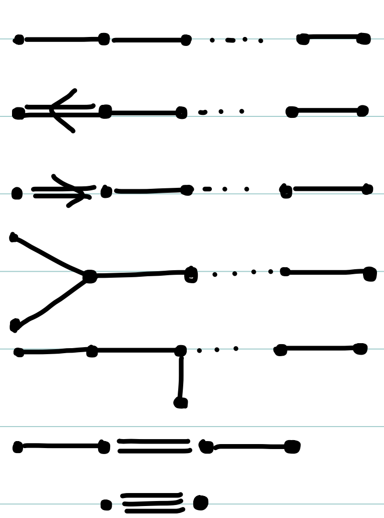
Prop: (W, S) is a Coxeter system

Ex: $G = GL_n$, $w \in S \iff w = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \longleftrightarrow (i, i+1) \in \mathfrak{S}_n$

W is the same for G, $[G,G]$ or $G/Z(G)$
GL_n SL_n PGL_n

$\rightsquigarrow$ need to add a dual pair of integral rep of W (root datum) to classify connected reductive groups

Classification of irreducible W



A_n GL_n, SL_n, PGL_n

B_n SO_{2n+1}, $Spin_{2n+1}$

C_n Sp_{2n}

D_n SO_{2n}, $Spin_{2n}$

E_n $n = 6, 7, 8$

F_4

G_2