$k = \mathbb{R}$ an alg. closed field

$R$ a fin. gen. $k$-algebra $\rightarrow \text{Spec}(R) = \{ \mathfrak{p} \in R \text{ a prime ideal} \}$

Zariski: $I$ an ideal

$\rightarrow \mathcal{V}(I) = \{ \mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p} \} \xrightarrow{\sim} \text{Spec}(R/I)$

give the closed sets of a topology

If $X = \text{Spec}(R)$ then $X(k) = \text{Hom}_{k-dgl}(R, k)$ are its $k$-rational points.

A morphism $\Phi : \text{Spec}(R) \rightarrow \text{Spec}(S)$ is given by

$\Phi(\mathfrak{p}) = \Phi^{-1}(\mathfrak{q})$

Where $\Phi : S \rightarrow R$ is a $k$-algebra homomorphism.

Example

$\mathbb{A}^n = \text{Spec}(k[t_1, \ldots, t_n])$

$k^n \xrightarrow{\sim} \mathbb{A}^n(k)$ where $\varepsilon_a : k[t_1, \ldots, t_n] \rightarrow k$

$i \mapsto \varepsilon_a$

Remark

(i) Assume $\{f_1, \ldots, f_n\} \subseteq R$ generate $R$ as a $k$-algebra then we have a surjective $k$-algebra homomorphism

$\Pi : k[t_1, \ldots, t_n] \rightarrow R$

Which gives a closed embedding $\text{Spec}(R) \hookrightarrow \mathbb{A}^n$, the image being $\mathcal{V}(\text{Ker}(\Pi))$.

(ii) $\text{Spec}(R_1) \times \text{Spec}(R_2) = \text{Spec}(R_1 \otimes_k R_2)$.

Definition

(i) If $R$ has no nilpotent elements then $\text{Spec}(R)$ is an affine variety.

(ii) A linear algebraic group (LAG) is an affine variety $G$ equipped with morphisms:

$\cdot m : G \times G \rightarrow G$ (multiplication)

$\cdot l : G \rightarrow G$ (inversion)
making \( G(k) \) a group.

**Examples**

(i) \( G_a = \text{Spec}(k[t]) \) is a group with

\[
\begin{align*}
m^* &: k[t] \to k[t] \otimes k[t] \\
t &= t \otimes 1 + 1 \otimes t
\end{align*}
\]

\[
\begin{align*}
v^* &: k[t] \to k[t] \\
t &= -t
\end{align*}
\]

\[
\varepsilon_{a,b} \circ m^* = \varepsilon_{a+b}
\]

\[
\varepsilon_a \circ v^* = \varepsilon_{-a}
\]

(ii) \( G_m = \text{Spec}(k[t,t^{-1}]) \) with

\[
\begin{align*}
m^*(t) &= t \otimes t \\
v^*(t) &= t^{-1}
\end{align*}
\]

(iii) \( GL_n = \text{Spec}(k[x_{ij}, \det^{-1}]) \)

**Exercise**

Write the mult. and inv. explicitly as algebra homomorphisms for \( GL_n \).

**Theorem**

Given a LAG \( G \) there exists a closed embedding \( G \hookrightarrow GL_n \).

**Consequence**

Any matrix \( g \in GL_n(k) \) has a product suzus decomposition where \( s \) is semisimple (diagonalisable) and \( u \) is unipotent. Same holds in \( G \).

\( \text{N.B.} \) if \( k = \mathbb{F}_p \) then every element in \( GL_n \) has finite order. We have \( g \) is semisimple if and only if it's \( p' \) and it's unipotent if and only if it's a \( p \)-element.
2) Global Structure

![Diagram](image)

**Connected Component**
- Minimal closed normal subgroup of finite index

**Radical**
- Maximal closed connected solvable normal subgroup

**Unipotent Radical**
- Maximal closed connected normal subgroup containing only unipotent elements

**Definition**

A LAG $G$ is:
- **connected** if $G = G^0$
- **reductive** if $R_u(G) = \{1\}$
- **semisimple** if $R(G) = \{1\}$
- **a torus** if $G \cong G_m \times \cdots \times G_m$

**Example**

$$G = G_{\text{Ln}} \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

$$R(G) = Z(G) \quad R(B) = B \quad R(T) = T$$

$$R_u(G) = \{1\} \quad R_u(B) = \left\{ \begin{pmatrix} 1 \\ * \end{pmatrix} \right\} \quad R_u(T) = \{1\}$$

N.B. $B \cong T \times R_u(B)$, which is true in general for connected solvable groups.
3) Remarkable Subgroups

Def: A torus of a LAG is a closed subgroup which is a torus.

Thm: Any two maximal tori of a LAG are conjugate.

Example

If \( G = GL_n \) then any subset \( SCG \) of commuting semisimple elements can be simultaneously diagonalised, i.e., there exists an element \( geG \) such that \( gSg^{-1} \subset \{ (*:0:*), (0:*:*), \ldots \} \).

Thm: For any LAG \( B \) in \( G \) the following hold:

(i) Any two Borel subgroups are \( G \)-conjugate.

(ii) Any maximal torus \( T \leq G \) is contained in a Borel subgroup \( BSG \). Moreover the pairs \( T \leq B \) are all \( G \)-conjugate.

(iii) If \( G \) is connected then \( N_G(B) = B \).

Example

If \( G = GL_n \) then by the Lie-Tolchin theorem if \( H \leq G \) is a connected solvable subgroup then there exists an element \( geG \) such that \( gHg^{-1} \subset \{ (*:*:0), (0:*:*), \ldots \} \).

4) Classification

Let \( G \) be a connected reductive algebraic group.

Thm: If \( T \leq G \) is a max. torus then \( C_G(T) = T \).
Def: Given $T$ a max torus we define

$$W = W_G(T) := N_G(T)/C_G(T) = N_G(T)/T$$

to be the Weyl group of $G$.

Example

$G = GL_n$, $T = \{(x, \cdots, x)\}$, $W_G(T) \cong S_n$ (symmetric group).

N.B: Here we have $N_G(T) = T \times W_G(T)$ but this is false in general. Try $SL_2$.

If $B \supset T$ is a Borel subgroup then we define

$$S = \{ w \in W \mid B \cup B w B \text{ is a subgroup of } G \}$$

the set of simple reflections of $W$.

Prop: $(W, S)$ is a Coxeter system.

Example

$G = GL_n$, $w \in S \iff W = \left( \begin{smallmatrix} 1 & & & & \\ & \ddots & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{smallmatrix} \right) \iff (i, i+1) \in S_n$.

Remark: The map $\bar{\tau} : [G, G] \rightarrow G \quad \tau \xrightarrow{\cdot x} xZ(G)$

maximal tori and $\bar{\tau}$ defines an isomorphism $W_{[G, G]}(T) \rightarrow W_G(\bar{\tau} T)$.

Similarly the map $\bar{\tau} : G \rightarrow G/Z(G)$ defines a bijection between maximal tori and $\bar{\tau}$ defines an isomorphism $W_G(T) \rightarrow W_{G/Z(G)}(\bar{\tau} T)$. 
Recall that \((W, s) = (W_1, s_1) \times \cdots \times (W_m, s_m)\) with \((W_i, s_i)\) irreducible.

These irreducible reflection groups are classified:

- \(A_n\) \(\bullet \quad \cdots \quad \bullet \)
- \(B_n\) \(\bullet \quad \leq \quad \cdots \quad \leq \quad \bullet \)
- \(C_n\) \(\bullet \quad \geq \quad \cdots \quad \geq \quad \bullet \)
- \(D_n\) \(\bullet \quad \vdash \quad \cdots \quad \vdash \quad \bullet \)
- \(E_n\) \(\bullet \quad \cdots \quad \bullet \quad (n \in \{6, 7, 8\})\)
- \(F_4\) \(\bullet \quad \cdots \quad \bullet \quad \cdots \quad \bullet \)
- \(G_2\) \(\bullet \quad \Xi \).