$k = \bar{k}$ an alg. closed field

R a fin. gen. k-algebra $\rightsquigarrow$ $\mathrm{Spec}(R) = \{\mathfrak{p} \subsetneq R \text{ a prime ideal}\}$

Zariski: $I \subseteq R$ an ideal

$\hookrightarrow V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\} \xleftrightarrow{1-1} \mathrm{Spec}(R/I)$

give the closed sets of a topology

If $X = \mathrm{Spec}(R)$ then $X(k) = \mathrm{Hom}_{k\text{-alg}}(R, k)$ are its k-rational points.

A morphism $\Phi : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is given by

$$\Phi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$$

Where $\varphi : S \to R$ is a k-algebra homomorphism.

Example

$\mathbb{A}^n = \mathrm{Spec}(k[t_1, \dots, t_n])$

$$\begin{array}{ccc} k^n & \xrightarrow{1-1} & \mathbb{A}^n(k) \\ a & \longmapsto & \varepsilon_a \end{array} \quad \text{where} \quad \begin{array}{ccc} \varepsilon_a : k[t_1, \dots, t_n] & \to & k \\ f & \mapsto & f(a) \end{array}.$$

Remark

(i) Assume $\{f_1, \dots, f_n\} \subseteq R$ generate R as a k-algebra then we have a surjective k-algebra homomorphism

$$\pi : k[t_1, \dots, t_n] \twoheadrightarrow R$$

Which gives a closed embedding $\mathrm{Spec}(R) \hookrightarrow \mathbb{A}^n$, the image being $V(\mathrm{Ker}(\pi))$.

(ii) $\mathrm{Spec}(R_1) \times \mathrm{Spec}(R_2) = \mathrm{Spec}(R_1 \otimes_k R_2)$.

Definition

(i) If R has no nilpotent elements then $\mathrm{Spec}(R)$ is an affine variety.

(ii) A linear algebraic group (LAG) is an affine variety G equipped with morphisms:

- $m : G \times G \to G$ (multiplication)
- $\iota : G \to G$ (inversion)

making $G(k)$ a group.

Examples

(i) $\mathbb{G}_a = \mathrm{Spec}(k[t])$ is a ~~group~~ LAG with

$$m^*: k[t] \longrightarrow k[t] \otimes k[t] \qquad\qquad \varepsilon_{(a,b)} \circ m^* = \varepsilon_{a+b}$$
$$t \longmapsto t \otimes 1 + 1 \otimes t$$

$$\iota^*: k[t] \longrightarrow k[t] \qquad\qquad \varepsilon_a \circ \iota^* = \varepsilon_{-a}$$
$$t \longmapsto -t$$

(ii) $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$ with

$$m^*(t) = t \otimes t$$
$$\iota^*(t) = t^{-1}$$

(iii) $GL_n = \mathrm{Spec}(k[x_{ij}, \det^{-1}])$

Exercise
Write the mult. and inv. explicitly as algebra homomorphisms for GL_n.

Theorem

Given a LAG G there exists a closed embedding $G \hookrightarrow GL_n$.

Consequence
Any matrix $g \in GL_n(k)$ has a product $su = us$ decomposition where s is semisimple (diagonalisable) and u is unipotent. Same holds in G.

⌈N.B: if $k = \overline{\mathbb{F}_p}$ then every element in ~~g~~ $G(k)$ has finite order. We have g is semisimp if and only if it's p' and it's unipotent if and only if it's a p-element.⌋

2) Global Structure

$$\begin{tikzcd} G \arrow[r, dash, "\text{finite}"] & G^\circ \arrow[r, dash, "\text{semisimple}"] & R(G) \arrow[r, dash, "\text{torus}"] & R_u(G) \arrow[r, dash, "\text{unipotent}"] & \{1\} \end{tikzcd}$$

Connected Component
- Minimal closed normal subgroup of finite index

Radical
- Maximal closed connected solvable normal subgroup

Unipotent Radical
- Maximal closed connected normal subgroup containing only unipotent elements.

Definition

A LAG G is:
- connected if $G = G^\circ$
- reductive if $R_u(G) = \{1\}$
- semisimple if $R(G) = \{1\}$
- a torus if $G \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

Example

$G = GL_n$ $\qquad B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ $\qquad T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

$R(G) = Z(G)$ $\qquad R(B) = B$ $\qquad R(T) = T$

$R_u(G) = \{1\}$ $\qquad R_u(B) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ $\qquad R_u(T) = \{1\}$

N.B. $B \cong T \ltimes R_u(B)$, which is true in general for connected solvable groups.

3) Remarkable Subgroups

Def: A torus of a LAG is a closed subgroup which is a torus.

Thm: Any two maximal tori of a LAG are conjugate.

Example

If $G = GL_n$ then any subset $S \subset G$ of commuting semisimple elements can be simultaneously diagonalised, i.e., there exists an element $g \in G$ such that $gSg^{-1} \subseteq \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

Thm: For any LAG ~~an~~ G the following hold:

(i) Any two Borel subgroups are G-conjugate.

(ii) Any maximal torus $T \leq G$ is contained in a Borel subgroup $B \leq G$. Moreover the pairs $T \leq B$ are all G-conjugate.

(iii) If G is connected then $N_G(B) = B$.

Example

~~By~~ If $G = GL_n$ then by the Lie-Kolchin theorem if $H \leq G$ is a connected solvable subgroup then there exists an element $g \in G$ such that $gHg^{-1} \leq B \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$.

4) Classification

Let G be a connected reductive algebraic group.

Thm: If $T \leq G$ is a max. torus then $C_G(T) = T$.

Def: Given T a max torus we define

$$W = W_G(T) := N_G(T)/C_G(T) = N_G(T)/T$$

to be "the" Weyl group of G.

Example

$G = GL_n$, $T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$, $W_G(T) \cong S_n$ (symmetric group).

N.B: Here we have $N_G(T) \cong T \rtimes W_G(T)$ but this is false in general. Try SL_2.

If $B \geq T$ is a Borel subgroup then we define

$$S = \{ w \in W \mid B \cup B\dot{w}B \text{ is a subgroup of } G \}$$

the set of simple reflections of W.

Prop: (W, S) is a Coxeter system.

Example

$G = GL_n$, $w \in S \iff w = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \longleftrightarrow (i, i+1) \in S_n$.

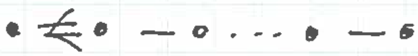
Remark: • The map $\pi : [G,G] \to G$, $x \mapsto xZ(G)$ defines a bijection between maximal tori and π defines an isomorphism $W_{[G,G]}(T) \to W_G(\pi(T))$.

• Similarly the map $\pi : G \to G/Z(G)$ defines a bijection between maximal tori and π defines an isomorphism $W_G(T) \to W_{\pi(G)}(\pi(T))$.

Recall that $(W,S) = (W_1,S_1) \times \cdots \times (W_r,S_r)$ with (W_i,S_i) irreducible

These irreducible reflection groups are classified.

A_n  GL_n, SL_n, PGL_n

B_n  SO_{2n+1}, $Spin_{2n+1}$

C_n  Sp_{2n}

D_n  SO_{2n}, $Spin_{2n}$

E_n $(n=6,7,8)$ 

F_4 

G_2